

5D SYM and 2D q -Deformed YM

YASUTAKA FUKUDA, TERUHIKO KAWANO, AND NARIAKI MATSUMIYA

Department of Physics, University of Tokyo, Hongo, Tokyo 113-0033, Japan

We study the AGT-like conjectured relation of a four-dimensional gauge theory on $S^3 \times S^1$ to two-dimensional q -deformed Yang-Mills theory on a Riemann surface Σ by using a five-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $S^3 \times \Sigma$, following the conjectured relation of the six-dimensional $\mathcal{N} = (2, 0)$ theory on S^1 to the five-dimensional Yang-Mills theory. Our results are in perfect agreement with both of the conjectures.

1 Introduction

The authors of the paper [1] have discussed that the partition function (the superconformal index) of a four-dimensional $\mathcal{N} = 2$ gauge theory on $S^3 \times S^1$ is given by two-dimensional q -deformed Yang-Mills theory on a Riemann surface Σ in the zero area limit. This is analogous to the Alday-Gaiotto-Tachikawa relation [2] of a four-dimensional $\mathcal{N} = 2$ gauge theory on S^4 and two-dimensional Liouville theory on a Riemann surface Σ . Following the prescription [3] of Gaiotto, the four-dimensional gauge theories are specified by the corresponding Riemann surface Σ , and it is widely thought that the putative six-dimensional $\mathcal{N} = (2, 0)$ theories on $S^3 \times S^1 \times \Sigma$ and $S^4 \times \Sigma$ underlie behind the former relation and the latter one, respectively.

However, the six-dimensional $\mathcal{N} = (2, 0)$ theory has not yet been formulated as a full-fledged theory. Therefore, one cannot use the $\mathcal{N} = (2, 0)$ theory to check the relations directly. But, it has been argued in [4, 5] that a five-dimensional $\mathcal{N} = 2$ Yang-Mills theory itself yields the full $\mathcal{N} = (2, 0)$ theory on S^1 . If this is the case, we can study the proposal of [1] more directly by using the five-dimensional $\mathcal{N} = 2$ Yang-Mills theory on $S^3 \times \Sigma$. In this paper, we will carry out the localization procedure in the five-dimensional theory to seek the relation of it with the two-dimensional q -deformed Yang-Mills theory on the Riemann surface.

In the previous paper [6], two of us have already studied the partition function of the five-dimensional $\mathcal{N} = 1$ Yang-Mills theory on $S^3 \times \mathbb{R}^2$ by using localization¹, and found that it yields two-dimensional bosonic Yang-Mills theory on \mathbb{R}^2 . On the flat \mathbb{R}^2 , it was not possible to distinguish between the ordinary Yang-Mills theory and the q -deformed one.

In this paper, therefore our previous results on the $S^3 \times \mathbb{R}^2$ in [6] will be extended to the $\mathcal{N} = 1$ theory on $S^3 \times \Sigma$ with Σ a closed Riemann surface. It will be further extended to the $\mathcal{N} = 2$ theory by introducing a hypermultiplet into the $\mathcal{N} = 1$ theory. We will

¹See [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] for recent related works on five-dimensional supersymmetric gauge theories.

compute the partition function in the resulting five-dimensional $\mathcal{N} = 2$ theory by carrying out the localization method, and will find that it is identical to the partition function of the two-dimensional q -deformed Yang-Mills theory. We will see that the parameter q derived from the five-dimensional theory is given in terms of the gauge coupling constant and the radius of the round S^3 , and it is in perfect agreement with the prediction from the conjecture of [4, 5] on the six-dimensional $\mathcal{N} = (2, 0)$ theory.

In the next section, we will give a very brief review on the quantization of the two-dimensional q -deformed Yang-Mills theory on a closed Riemann surface Σ , and a five-dimensional supersymmetric Yang-Mills theory on a flat Euclidean space \mathbb{R}^5 will be given in section 3.

In order to put the five-dimensional theory on $S^3 \times \Sigma$, we will explain the Killing spinors on the round S^3 and the “partial twisting” [17, 18] on the surface Σ to define a supersymmetry transformation on the $S^3 \times \Sigma$ in section 4, and give the supersymmetry transformations, their algebra, and the Lagrangian on the $S^3 \times \Sigma$ in section 5.

In section 6, we will carry out the localization procedure to compute the partition function of the five-dimensional theory. It will turn out that the result of summing up the one-loop determinants yields the partition function of the two-dimensional q -deformed Yang-Mills theory, but not of the two-dimensional ordinary Yang-Mills theory.

In section 7, we will discuss that the parameter q calculated in the five-dimensional theory is in perfect agreement with the prediction of the conjecture [4, 5] for the six-dimensional $\mathcal{N} = (2, 0)$ theory.

In appendix A, our notations on the gamma matrices will be explained. Since we will need the Clebsch-Gordan coefficients in the one-loop calculation in section 6, the Clebsch-Gordan coefficients necessary for the calculations are listed in appendix B.

2 Two-Dimensional q -Deformed Yang-Mills Theory on Σ

We begin with a very brief review on two-dimensional Yang-Mills theory on a closed Riemann surface Σ . For more details, see [19, 20]. The two-dimensional Yang-Mills theory with the gauge group² G on the Riemann surface Σ is described by gauge fields $v_z, v_{\bar{z}}$ with the Lagrangian³

$$\text{tr} \left[(g^{\bar{z}z} v_{\bar{z}z})^2 \right],$$

where the field strength $v_{\bar{z}z}$ is defined by $\partial_{\bar{z}} v_z - \partial_z v_{\bar{z}} + ig [v_{\bar{z}}, v_z]$, and $g^{\bar{z}z}$ is the Kähler metric on Σ .

Introducing a scalar field ϕ in the adjoint representation of G , one may rewrite the Lagrangian into

$$\mathcal{L}_{YM} = \text{tr} [\phi^2 - 2i\phi g^{\bar{z}z} v_{\bar{z}z}].$$

Let us consider the quantization of this system by the path integral. The first term in the Lagrangian will be treated as a perturbation, and the second term is the bosonic part

²We assume that the gauge group G is simple.

³The Lagrangian is the conventional Yang-Mills Lagrangian multiplied by a factor two, in our convention.

of the Lagrangian of a topological field theory. We will first perform the path integral over the gauge fields. To this end, by using a gauge transformation

$$\phi \rightarrow \phi + ig [\omega(z, \bar{z}), \phi], \quad v_z \rightarrow v_z - D_z \omega(z, \bar{z}),$$

we impose the gauge fixing condition on the scalar field ϕ ,

$$\phi(z, \bar{z}) = \sum_{i=1}^r \phi^i H_i, \quad (1)$$

where H_i ($i = 1, \dots, r$) are the generators of the Cartan subalgebra of G of rank r . Integrating over the Cartan part of the fluctuations

$$\tilde{v}_z(z, \bar{z}) = \sum_{i=1}^r \tilde{v}_z^i(z, \bar{z}) H_i,$$

of the gauge fields, one obtains the delta functions imposing the conditions

$$\partial_z \phi^i = 0, \quad \partial_{\bar{z}} \phi^i = 0,$$

on ϕ , and they require that ϕ^i should be a constant. Then, the remaining fluctuations \tilde{v}_z , $\tilde{v}_{\bar{z}}$ of the gauge fields, which should be integrated about the classical solution $\phi^i = \text{const.}$, are

$$\tilde{v}_z = \sum_{\alpha \in \Lambda} \tilde{v}_z^\alpha(z, \bar{z}) E_\alpha,$$

where Λ is the set of all the roots of the Lie algebra of G , and the root generators E_α satisfy the algebra

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \sum_{i=1}^r \alpha_i H_i \equiv \alpha \cdot H,$$

with the normalization

$$\text{tr}[H^i H^j] = \delta^{ij}, \quad \text{tr}[E_{-\alpha} E_\alpha] = 1.$$

Therefore, the Lagrangian \mathcal{L}_{YM} gives

$$\mathcal{L}_{\text{cl}} - \sum_{\alpha \in \Lambda} 2g (\alpha \cdot \phi) g^{\bar{z}z} \tilde{v}_{\bar{z}}^{-\alpha} \tilde{v}_z^\alpha,$$

with $(\alpha \cdot \phi) = \sum_{i=1}^r \alpha^i \phi^i$, where \mathcal{L}_{cl} is the classical value of the Lagrangian \mathcal{L}_{YM} given by

$$\sum_{i=1}^r [\phi^i \phi^i - 2ig^{\bar{z}z} v_{\bar{z}z}^i \phi^i],$$

with the non-trivial first Chern class $\int_{\Sigma} v_{\bar{z}z}^i dz \wedge d\bar{z} \neq 0$.

We will add the gauge-fixing term and the ghost term

$$\sum_{\alpha \in \Lambda} [b^{-\alpha} \phi^\alpha - 2g (\alpha \cdot \phi) \bar{c}^{-\alpha} c^\alpha],$$

with the auxiliary fields and the ghosts

$$b(z, \bar{z}) = \sum_{\alpha \in \Lambda} b^\alpha(z, \bar{z}) E_\alpha, \quad c(z, \bar{z}) = \sum_{\alpha \in \Lambda} c^\alpha(z, \bar{z}) E_\alpha, \quad \bar{c}(z, \bar{z}) = \sum_{\alpha \in \Lambda} \bar{c}^\alpha(z, \bar{z}) E_\alpha,$$

to impose the gauge fixing condition (1) correctly.

The integration over the fields b^α and ϕ^α is trivial, while the integration over the fluctuation of the gauge fields and the pair of ghosts gives

$$\prod_{\alpha \in \Lambda} \frac{\text{Det}_{(0,0)} [2g(\alpha \cdot \phi)]}{\text{Det}_{(1,0)} [2g(\alpha \cdot \phi)]},$$

where $\text{Det}_{(p,q)} [D]$ is the functional determinant of the operator D over the space of the (p, q) -forms on the Riemann surface Σ . As explained in [19], due to the Hodge decomposition, it yields

$$\prod_{\alpha \in \Lambda} [2g(\alpha \cdot \phi)]^{\chi(\Sigma)/2} = \prod_{\alpha \in \Lambda_+} [2ig(\alpha \cdot \phi)]^{\chi(\Sigma)}, \quad (2)$$

up to an overall constant, with Λ_+ the set of the positive roots, where $\chi(\Sigma)$ is the Euler character of Σ .

Let us proceed to the q -deformed Yang-Mills theory by making the scalar ϕ^i periodic. Following [20], we will use the method of images to extend (2) to this case. For brevity, we will take the $SU(2)$ gauge group, and the measure (2) is replaced via the method of images by

$$\prod_{n=-\infty}^{\infty} \left(2\sqrt{2}ig\phi + i\frac{n}{l} \right)^{\chi(\Sigma)} = \left[2i \sin \left(2\sqrt{2}\pi gl\phi \right) \right]^{\chi(\Sigma)}, \quad (3)$$

for the periodicity $\phi \rightarrow \phi + n/(2\sqrt{2}gl)$, with the abbreviation $\phi = \phi^1$ here. Note that the zeta regularization has been used.

Since the classical Lagrangian \mathcal{L}_{cl} contains the term $-4\sqrt{2}\pi m\phi/g$ from the non-trivial flux

$$\frac{1}{2\pi} \int_{\Sigma} v_{\bar{z}z} d\bar{z} \wedge dz = \frac{\sqrt{2}}{g} m, \quad (m \in \mathbb{Z})$$

whose normalization will be explained in section 7, summing over all the fluxes m reduces the integration over ϕ into the summation over $\phi = ing/(2\sqrt{2})$ with $n \in \mathbb{Z}$. Therefore, at each point $\phi = ing/(2\sqrt{2})$, the measure (3) yields

$$[2i \sin(\pi ig^2 ln)]^{\chi(\Sigma)} = [n]_q^{\chi(\Sigma)},$$

with $q = \exp(-2\pi g^2 l)$, where $[x]_q = (q^{x/2} - q^{-x/2})$. Naïvely speaking, this is why this theory is called the q -deformed theory.

3 Five-Dimensional Super Yang-Mills Theory on \mathbb{R}^5

Let us proceed to a brief explanation about a five-dimensional supersymmetric Yang-Mills theory on \mathbb{R}^5 , which can be obtained by the dimensional reduction in the time direction of six-dimensional maximally supersymmetric Yang-Mills theory in a flat Minkowski space.

In terms of five-dimensional $\mathcal{N} = 1$ supermultiplets, the vector multiplet in the $\mathcal{N} = 2$ theory consists of an $\mathcal{N} = 1$ vector multiplet and an $\mathcal{N} = 1$ hypermultiplet in the adjoint representation of the gauge group G .

The vector multiplet consists of a gauge field v_M ($M = 1, \dots, 5$), a real scalar field σ , auxiliary fields $D^{\dot{\alpha}}_{\dot{\beta}}$, and a spinor field $\Psi^{\dot{\alpha}}$, where the indices $\dot{\alpha}, \dot{\beta}$ label the components of the fundamental representation **2** of $SU(2)$ R -symmetry⁴. The spinor field $\Psi^{\dot{\alpha}}$ obeys the symplectic Majorana condition

$$(\Psi^{\dot{\beta}})^T C_5 \epsilon_{\dot{\beta}\dot{\alpha}} = (\Psi_{\dot{\alpha}})^{\dagger} \equiv \bar{\Psi}_{\dot{\alpha}},$$

where T denotes the transpose, and $\epsilon_{\dot{\alpha}\dot{\beta}}$ is the invariant tensor of the $SU(2)$ R -symmetry. The auxiliary fields $D^{\dot{\alpha}}_{\dot{\beta}}$ are anti-Hermitian and in the adjoint representation of the $SU(2)$ R -symmetry;

$$D^{\dot{\alpha}}_{\dot{\beta}} = -(D^{\dot{\beta}}_{\dot{\alpha}})^{\dagger}, \quad D^{\dot{\alpha}}_{\dot{\gamma}} \epsilon^{\dot{\gamma}\dot{\beta}} = D^{\dot{\beta}}_{\dot{\gamma}} \epsilon^{\dot{\gamma}\dot{\alpha}}, \quad D^{\dot{\alpha}}_{\dot{\alpha}} = 0.$$

Our notations for the charge conjugation matrix C_5 and the gamma matrices Γ^M are explained in appendix A. Since all the fields are in the adjoint representation of the gauge group G , they are denoted in the matrix notation as

$$\Phi = \Phi^A T^A$$

with the normalization⁵ $\text{tr}[T^A T^B] = \delta^{AB}$.

On a flat Euclidean space \mathbb{R}^5 , the Lagrangian $\mathcal{L}^{(0)}_V$ of the vector multiplet is given⁶ by

$$\text{tr} \left[-\frac{1}{4} v_{MN} v^{MN} + \frac{1}{2} D_M \sigma D^M \sigma + i \bar{\Psi}_{\dot{\alpha}} \Gamma^M D_M \Psi^{\dot{\alpha}} - g \bar{\Psi}_{\dot{\alpha}} [\sigma, \Psi^{\dot{\alpha}}] - \frac{1}{4} D^{\dot{\alpha}}_{\dot{\beta}} D^{\dot{\beta}}_{\dot{\alpha}} \right], \quad (4)$$

where v_{MN} is the field strength

$$v_{MN} = \partial_M v_N - \partial_N v_M + ig [v_M, v_N],$$

of the gauge field v_M , and the covariant derivatives $D_M \Phi$ is given by

$$D_M \Phi = \partial_M \Phi + ig [v_M, \Phi].$$

The Lagrangian $\mathcal{L}^{(0)}_V$ is left invariant under a supersymmetry transformation

$$\begin{aligned} \delta_{\Sigma}^{(0)} v_M &= -i \bar{\Sigma}_{\dot{\alpha}} \Gamma_M \Psi^{\dot{\alpha}}, & \delta_{\Sigma}^{(0)} \sigma &= i \bar{\Sigma}_{\dot{\alpha}} \Psi^{\dot{\alpha}}, \\ \delta_{\Sigma}^{(0)} \Psi^{\dot{\alpha}} &= -\frac{1}{2} \left(\frac{1}{2} v_{MN} \Gamma^{MN} \Sigma^{\dot{\alpha}} + \Gamma^M D_M \sigma \Sigma^{\dot{\alpha}} + D^{\dot{\alpha}}_{\dot{\beta}} \Sigma^{\dot{\beta}} \right), \\ \delta_{\Sigma}^{(0)} D^{\dot{\alpha}}_{\dot{\beta}} &= i [D_M \bar{\Psi}_{\dot{\beta}} \Gamma^M \Sigma^{\dot{\alpha}} + \bar{\Sigma}_{\dot{\beta}} \Gamma^M D_M \Psi^{\dot{\alpha}} + ig ([\sigma, \bar{\Psi}_{\dot{\beta}}] \Sigma^{\dot{\alpha}} + \bar{\Sigma}_{\dot{\beta}} [\sigma, \Psi^{\dot{\alpha}}])], \end{aligned} \quad (5)$$

⁴Although the R -symmetry of the $\mathcal{N} = 2$ theory is $SO(5)$, its subgroup $SO(4) \simeq SU(2) \times SU(2)$ is manifest in terms of the $\mathcal{N} = 1$ supermultiplets, and we will respect only one of the two $SU(2)$ s, in this paper.

⁵The gauge group G is assumed to be simple.

⁶The sign of the Lagrangian $\mathcal{L}^{(0)}_V$ is opposite to the one in the previous paper [6].

where the transformation parameter $\Sigma^{\dot{\alpha}}$ is also a symplectic Majorana spinor;

$$\bar{\Sigma}_{\dot{\alpha}} = (\Sigma^{\dot{\beta}})^T C_5 \epsilon_{\dot{\beta}\dot{\alpha}}.$$

The hypermultiplet consists of complex scalar fields $H_{\dot{\alpha}}$, a spinor field Ξ , and auxiliary fields $F_{H\alpha}$ ($\alpha = 1, 2$), where the index α is distinct from the one $\dot{\alpha}$ of the $SU(2)_R$ symmetry. Since they all transform in the adjoint representation under a gauge transformation, they are also denoted in the matrix notation.

They have the free Lagrangian $\mathcal{L}^{(0)}_H$ and the interaction $\mathcal{L}^{(0)}_{\text{int}}$ with the vector multiplet, and they are given by

$$\mathcal{L}^{(0)}_H = \text{tr} \left[-D^M \bar{H}^{\dot{\alpha}} D_M H_{\dot{\alpha}} - i \bar{\Xi} \Gamma^M D_M \Xi + \bar{F}_H{}^{\alpha} F_{H\alpha} \right], \quad (6)$$

$$\begin{aligned} \mathcal{L}^{(0)}_{\text{int}} = \text{tr} \Bigg[& g^2 [\sigma, \bar{H}^{\dot{\alpha}}] [\sigma, H_{\dot{\alpha}}] + ig D^{\dot{\alpha}}{}_{\dot{\beta}} [\bar{H}^{\dot{\beta}}, H_{\dot{\alpha}}] - g \bar{\Xi} [\sigma, \Xi] \\ & - 2g \bar{\Xi} [H_{\dot{\alpha}}, \Psi^{\dot{\alpha}}] + 2g [\bar{H}^{\dot{\alpha}}, \bar{\Psi}_{\dot{\alpha}}] \Xi \Bigg], \end{aligned} \quad (7)$$

where $\bar{H}^{\dot{\alpha}}$ is the complex conjugate of $H_{\dot{\alpha}}$, and $\bar{\Xi} = \Xi^{\dagger}$. They are left invariant under a supersymmetry transformation [8]

$$\begin{aligned} \delta_{\Sigma}^{(0)} H_{\dot{\alpha}} &= -i \bar{\Sigma}_{\dot{\alpha}} \Xi, \\ \delta_{\Sigma}^{(0)} \Xi &= (\Gamma^M D_M H_{\dot{\alpha}} + ig [\sigma, H_{\dot{\alpha}}]) + F_{H\alpha} \check{\Sigma}^{\alpha}, \\ \delta_{\Sigma}^{(0)} F_{H\alpha} &= i \bar{\Sigma}_{\alpha} \left[\Gamma^M D_M \Xi - ig [\sigma, \Xi] - 2ig [H_{\dot{\beta}}, \Psi^{\dot{\beta}}] \right], \end{aligned} \quad (8)$$

if accompanied by the transformation (5). The transformation parameters $\check{\Sigma}^{\alpha}$ are linearly independent spinors of $\Sigma^{\dot{\alpha}}$ and also obey the symplectic-Majorana condition

$$\bar{\Sigma}_{\alpha} = (\check{\Sigma}^{\beta})^T C_5 \epsilon_{\beta\alpha}.$$

The supersymmetry transformation (5),(8) yields a closed algebra on any field Φ in the vector multiplet and the hypermultiplet for the supersymmetry parameters $\Theta^{\dot{\alpha}}, \Sigma^{\dot{\alpha}}$, specified in section 4 as

$$\left[\delta_{\Theta}^{(0)}, \delta_{\Sigma}^{(0)} \right] \Phi = \xi^M \partial_M \Phi + \delta_G \Phi \equiv \delta^{(0)} \Phi,$$

with

$$\xi^M = i \bar{\Theta}_{\dot{\gamma}} \Gamma^M \Sigma^{\dot{\gamma}},$$

where δ_G is a gauge transformation with the parameter

$$\omega = i \left[\bar{\Theta}_{\dot{\gamma}} \Gamma^M \Sigma^{\dot{\gamma}} v_M + \bar{\Theta}_{\dot{\gamma}} \Sigma^{\dot{\gamma}} \sigma \right], \quad (9)$$

and therefore, on an adjoint field Φ ,

$$\delta_G \Phi = ig [\omega, \Phi].$$

4 SUSY Parameters on $S^3 \times \Sigma$

In the previous paper [6], we have considered the $\mathcal{N} = 1$ supersymmetric Yang-Mills theory on $S^3 \times \mathbb{R}^2$, where we have picked up the Killing spinor ϵ on the unit round S^3 obeying the Killing spinor equation

$$\nabla_m \epsilon = \frac{i}{2} \gamma_m \epsilon, \quad (m = 1, 2, 3) \quad (10)$$

with the spin connection ω_m^{ab} ($a, b = 1, 2, 3$) of the unit round S^3 in the covariant derivative

$$\nabla_m \epsilon = \partial_m \epsilon + \frac{1}{4} \omega_m^{ab} \gamma^{ab} \epsilon,$$

and have formed the supersymmetry transformation parameter $\Sigma^{\dot{\alpha}}$ ($\dot{\alpha} = 1, 2$) given by

$$\Sigma^{\dot{\alpha}=1} = \epsilon \otimes \zeta_+, \quad \Sigma^{\dot{\alpha}=2} = C_3^{-1} \epsilon^* \otimes \zeta_-, \quad (11)$$

where $*$ denotes the complex conjugation, and two-dimensional Weyl spinors⁷

$$\zeta_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix},$$

on \mathbb{R}^2 are the eigenvectors of $i\Gamma^4\Gamma^5$; $i\Gamma^4\Gamma^5\zeta_{\pm} = \pm\zeta_{\pm}$.

One of important properties of the parameter $\Sigma^{\dot{\alpha}}$ is that it obeys the condition

$$\Gamma^{45} \Sigma^{\dot{\alpha}} = -2i N^{\dot{\alpha}}_{\dot{\beta}} \Sigma^{\dot{\beta}} \equiv -2\tilde{\Sigma}^{\dot{\alpha}},$$

where the matrix $N^{\dot{\alpha}}_{\dot{\beta}}$ is defined by

$$(N^{\dot{\alpha}}_{\dot{\beta}}) = \frac{1}{2} \sigma_3,$$

and another condition

$$\nabla_m \Sigma^{\dot{\alpha}} = i N^{\dot{\alpha}}_{\dot{\beta}} \Gamma_m \Sigma^{\dot{\beta}} = \Gamma_m \tilde{\Sigma}^{\dot{\beta}},$$

is a direct consequence of the Killing spinor equation (10) of ϵ .

The linearly independent parameter $\tilde{\Sigma}^{\alpha}$ is then given by

$$\tilde{\Sigma}^{\alpha=1} = \epsilon \otimes \zeta_-, \quad \tilde{\Sigma}^{\alpha=2} = C_3^{-1} \epsilon^* \otimes \zeta_+, \quad (12)$$

and obeys the similar conditions

$$\Gamma^{45} \tilde{\Sigma}^{\alpha} = 2i N^{\alpha}_{\beta} \tilde{\Sigma}^{\beta} \equiv -2\tilde{\Sigma}^{\alpha}, \quad \nabla_m \tilde{\Sigma}^{\alpha} = -i N^{\alpha}_{\beta} \Gamma_m \tilde{\Sigma}^{\beta} = \Gamma_m \tilde{\Sigma}^{\alpha},$$

where

$$(N^{\alpha}_{\beta}) = \frac{1}{2} \sigma_3.$$

⁷They were denoted as χ_{\pm} in the previous paper [6].

When we replace \mathbb{R}^2 in $S^3 \times \mathbb{R}^2$ by a Riemann surface Σ , we would like to keep using $\Sigma^\dot{\alpha}$ as a supersymmetry transformation parameter on $S^3 \times \Sigma$. To this end, one needs to introduce a background gauge field by gauging the $SU(2)_R$ symmetry. Suppose that the covariant derivative on a Weyl spinor ξ_+ of positive chirality on the Riemann surface Σ is given by

$$\nabla_z \xi_+ = \left(\partial_z - \frac{i}{2} \omega_z \right) \xi_+,$$

with the spin connection ω_z on Σ , and then the gauging of the $SU(2)_R$ symmetry yields the covariant derivative

$$\nabla_z \Sigma^\dot{\alpha} = \partial_z \Sigma^\dot{\alpha} + \frac{1}{2} \omega_z \Gamma^{45} \Sigma^\dot{\alpha} + i A_{z\dot{\beta}}^\dot{\alpha} \Sigma^\dot{\beta} = \partial_z \Sigma^\dot{\alpha} + i (A_{z\dot{\beta}}^\dot{\alpha} - \omega_z N_{\dot{\beta}}^\dot{\alpha}) \Sigma^\dot{\beta} \quad (13)$$

on the parameter $\Sigma^\dot{\alpha}$. We here have identified the local complex coordinates z, \bar{z} on Σ with $z = x^4 + ix^5$, $\bar{z} = x^4 - ix^5$. If one takes the background gauge field $A_{z\dot{\beta}}^\dot{\alpha}$ as

$$A_{z\dot{\beta}}^\dot{\alpha} = \omega_z N_{\dot{\beta}}^\dot{\alpha},$$

it is obvious from (13) that the parameter $\Sigma^\dot{\alpha}$ in (11) yields a covariantly constant spinor on Σ so that $\Sigma^\dot{\alpha}$ is well-defined on the Riemann surface Σ .

As for the other parameter $\check{\Sigma}^\alpha$, we will introduce a background gauge field $\check{A}_{z\beta}^\alpha = -\omega_z N_{\beta}^\alpha$ to make $\check{\Sigma}^\alpha$ covariantly constant and so well-defined on Σ .

This ‘‘partial twisting’’ [17, 18] affects the spin representations of the fields carrying the indices of the $SU(2)_R$ symmetry or the index α of another $SU(2)$ symmetry. In order to see this and also for later use, it is convenient to give the gauge field v^M and the spinor $\Psi^\dot{\alpha}$ in the vector multiplet in terms of three-dimensional tensors and spinors as

$$\begin{aligned} v^m \quad (m = 1, 2, 3), \quad v_z &= \frac{1}{2} (v_4 - iv_5), \quad v_{\bar{z}} = \frac{1}{2} (v_4 + iv_5), \\ \Psi^{\dot{\alpha}=1} &= \lambda \otimes \zeta_+ + \psi \otimes \zeta_-, \quad \Psi^{\dot{\alpha}=2} = C_3^{-1} \psi^* \otimes \zeta_+ + C_3^{-1} \lambda^* \otimes \zeta_-, \\ D &= D^1_1 + g^{\bar{z}z} v_{\bar{z}z}, \quad F_z = \frac{1}{2} D^1_2, \quad \bar{F}_{\bar{z}} = \frac{1}{2} D^2_1. \end{aligned}$$

As for the hypermultiplet,

$$\begin{aligned} (H_{\dot{\alpha}}) &= \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} \tilde{H} \\ (H)^* \end{pmatrix}, \\ \Xi &= \tilde{\chi} \otimes \zeta_+ + C_3^{-1} (\chi)^* \otimes \zeta_-. \end{aligned}$$

In terms of these fields, after the partial twisting, in the vector multiplet, λ becomes a scalar on Σ , while ψ becomes a $(1, 0)$ -form ψ_z . The auxiliary fields D and F_z are a scalar and a $(1, 0)$ -form, respectively. In the hypermultiplet, the scalars \tilde{H} , H become Weyl spinors of positive chirality, while $\tilde{\chi}$, χ are unaffected to remain Weyl spinors of positive chirality. The auxiliary fields F_{H1} and F_{H2} become Weyl spinors of negative chirality and positive chirality, respectively.

5 Supersymmetry on $S^3 \times \Sigma$

When going onto the $S^3 \times \Sigma$, we turn on the spin connections of $S^3 \times \Sigma$ and the background gauge fields $A^{\dot{\alpha}}_{\dot{\beta}}$, A^{α}_{β} in the covariant derivatives in the supersymmetry transformation (5), (8). However, the supersymmetry transformation (5), (8) no longer yields a closed algebra, since the supersymmetry parameters $\Sigma^{\dot{\alpha}}$, $\tilde{\Sigma}^{\alpha}$ aren't covariantly constant along the S^3 , although the gauging of the $SU(2)_R$ symmetry and the other $SU(2)$ symmetry makes them covariantly constant along the Riemann surface Σ .

In order to give a closed algebra of the supersymmetry transformations on $S^3 \times \Sigma$, one needs to modify the transformations by adding the terms

$$\delta'_{\Sigma} D^{\dot{\alpha}}_{\dot{\beta}} = -2i \left(\tilde{\Sigma}_{\dot{\beta}} \Psi^{\dot{\alpha}} + \bar{\Psi}_{\dot{\beta}} \tilde{\Sigma}^{\dot{\alpha}} \right),$$

to (5) for the vector multiplet and

$$\delta'_{\Sigma} \Xi = 2H_{\dot{\alpha}} \tilde{\Sigma}^{\dot{\alpha}}, \quad \delta'_{\Sigma} F_{\alpha} = \frac{i}{2} \tilde{\Sigma}_{\alpha} \Gamma^{45} \Xi,$$

to (8) for the hypermultiplet.

The modified transformation $\delta_{\Sigma} = \delta_{\Sigma}^{(0)} + \delta'_{\Sigma}$ indeed gives the closed supersymmetry algebra on any field Φ in the vector multiplet and the hypermultiplet as

$$[\delta_{\Theta}, \delta_{\Sigma}] \Phi = \mathcal{L}_{\xi} \Phi + \delta_G \Phi + \delta_R \Phi,$$

with the parameters (11),(12), and their analogues $\Theta^{\dot{\alpha}}$, $\tilde{\Theta}^{\alpha}$ where ϵ is replaced by a solution η to (10) in (11),(12), respectively. The transformation δ_G is the same gauge transformation as before with the parameter (9), and δ_R is the transformation of a $U(1)$ subgroup of the $SU(2)_R$ symmetry and the other $SU(2)$ symmetry and is given by

$$\delta_R \Phi^{\dot{\alpha}} = 2i \left(\tilde{\Theta}_{\dot{\beta}} \Sigma^{\dot{\alpha}} - \tilde{\Sigma}_{\dot{\beta}} \Theta^{\dot{\alpha}} \right) \Phi^{\dot{\beta}}, \quad \delta_R \Phi_{\alpha} = -2i \Phi_{\beta} \left(\tilde{\Theta}_{\alpha} \tilde{\Sigma}^{\beta} - \tilde{\Sigma}_{\alpha} \tilde{\Theta}^{\beta} \right),$$

for the fundamental representation $\Phi^{\dot{\alpha}}$ of the $SU(2)_R$ and the anti-fundamental representation Φ_{α} of the other $SU(2)$. The partial derivative which was in the translation is replaced by the Lie derivative \mathcal{L}_{ξ} with respect to the vector $\xi^M = i\tilde{\Theta}_{\dot{\gamma}} \Gamma^M \Sigma^{\dot{\gamma}}$. Here we have defined the Lie derivative \mathcal{L}_{ξ} on a five-dimensional spinor field Ψ as

$$\mathcal{L}_{\xi} \Psi = \xi^M \nabla_M \Psi + \frac{1}{4} (\nabla_M \xi_N) \Gamma^{MN} \Psi.$$

Note that the translation in \mathbb{R}^5 is extended to the infinitesimal diffeomorphism with the parameter ξ^M on the curved space $S^3 \times \Sigma$, and the diffeomorphism also transforms the background vielbein non-trivially. Therefore, the Lorentz transformation in the second term of the above definition is needed to compensate the diffeomorphism in order to keep the background vielbein intact.

In terms of the three-dimensional fields, the supersymmetry transformation yields

$$\begin{aligned}
\delta_\Sigma v_m &= -i [\bar{\epsilon} \gamma_m \lambda - \bar{\lambda} \gamma_m \epsilon], & \delta_\Sigma v_z &= -\bar{\epsilon} \psi_z, & \delta_\Sigma \sigma &= i [\bar{\epsilon} \lambda - \bar{\lambda} \epsilon], \\
\delta_\Sigma \lambda &= -\frac{1}{2} \left[\frac{1}{2} v_{mn} \gamma^{mn} + \gamma^m D_m \sigma + D \right] \epsilon, \\
\delta_\Sigma \psi_z &= [i v_{mz} \gamma^m \epsilon - i D_z \sigma \epsilon - F_z C_3^{-1} \epsilon^*], \\
\delta_\Sigma D &= i \left[D_m \bar{\lambda} \gamma^m \epsilon + \bar{\epsilon} \gamma^m D_m \lambda + i g ([\sigma, \bar{\lambda}] \epsilon + \bar{\epsilon} [\sigma, \lambda]) + \frac{i}{2} (\bar{\epsilon} \lambda - \bar{\lambda} \epsilon) \right], \\
\delta_\Sigma F_z &= i \left[-\epsilon^T C_3 \gamma^m D_m \psi_z + 2i \epsilon^T C_3 D_z \lambda + i g \epsilon^T C_3 [\sigma, \psi_z] - \frac{i}{2} \epsilon^T C_3 \psi_z \right],
\end{aligned} \tag{14}$$

for the vector multiplet, and

$$\begin{aligned}
\delta_\Sigma \tilde{H} &= -i \bar{\epsilon} \tilde{\chi}, & \delta_\Sigma H &= -i \bar{\epsilon} \chi, \\
\delta_\Sigma \tilde{\chi} &= \left[D_m \tilde{H} \gamma^m + i g [\sigma, \tilde{H}] + i \tilde{H} \right] \epsilon + [-2i (D_z H)^* + F_{H2}] C_3^{-1} \epsilon^*, \\
\delta_\Sigma \chi &= [D_m H \gamma^m + i g [\sigma, H] + i H] \epsilon + \left[2i (D_z \tilde{H})^* - (F_{H1})^* \right] C_3^{-1} \epsilon^*, \\
\delta_\Sigma F_{H1}^* &= i \left[(D_m \chi)^T C_3 \gamma^m + 2i (D_z \tilde{\chi})^\dagger + \frac{i}{2} \chi^T C_3 + i g [\sigma, \chi^T] C_3 \right. \\
&\quad \left. + 2i g ([\tilde{H}^*, \bar{\psi}] + [H, \lambda^T] C_3) \right] \epsilon, \\
\delta_\Sigma F_{H2}^* &= -i \left[(D_m \tilde{\chi})^\dagger \gamma^m + 2i (D_z \chi)^T C_3 + \frac{i}{2} \tilde{\chi}^\dagger - i g [\sigma, \tilde{\chi}^\dagger] \right. \\
&\quad \left. - 2i g ([\tilde{H}^*, \bar{\lambda}] + [H, \psi^T] C_3) \right] C_3^{-1} \epsilon^*,
\end{aligned} \tag{15}$$

for the hypermultiplet.

The Lagrangians $\mathcal{L}_V^{(0)}$, $\mathcal{L}_H^{(0)}$, $\mathcal{L}_{\text{int}}^{(0)}$ in the \mathbb{R}^5 need to be covariantized in order to put them on the curved space $S^3 \times \Sigma$. In addition, the gauging of $SU(2)_R$ symmetry and the other $SU(2)$ symmetry in them will be done to consider the supersymmetry transformation (14), (15) with the parameters (11), (12) of the Lagrangians.

However, these aren't enough to obtain invariant Lagrangians under the transformation (14), (15), and to this end, one needs additional terms to the covariantized Lagrangians. In fact, it turns out that the additional terms to $\mathcal{L}_V^{(0)}$ of the vector multiplet are given by

$$\begin{aligned}
\mathcal{L}'_V &= \text{tr} \left[N^{\dot{\alpha}}_{\dot{\beta}} \bar{\Psi}_{\dot{\alpha}} \Psi^{\dot{\alpha}} + \left(i N^{\dot{\alpha}}_{\dot{\beta}} D^{\dot{\beta}}_{\dot{\alpha}} - \frac{1}{2} \varepsilon^{ij} v_{ij} \right) \sigma + \sigma^2 + \frac{1}{2} \omega_{\text{c.s.}} \right] \\
&= \text{tr} \left[\frac{1}{2} g^{\bar{z}z} \bar{\psi}_{\bar{z}} \psi_z + \bar{\lambda} \lambda + \sigma \sigma + i \sigma (D - 2g^{\bar{z}z} v_{\bar{z}z}) + \frac{1}{2} \omega_{\text{c.s.}} \right],
\end{aligned}$$

with the Chern-Simons term

$$\omega_{\text{c.s.}} = \varepsilon^{mnk} \left(v_m \partial_n v_k + \frac{i}{3} g v_m [v_n, v_k] \right),$$

and to $\mathcal{L}_H^{(0)}$ of the hypermultiplet,

$$\mathcal{L}'_H = \text{tr} \left[-\frac{i}{2} \Xi \Gamma^{45} \Xi - \bar{H}^\alpha H_{\dot{\alpha}} \right] = -\text{tr} \left[\frac{1}{2} (\tilde{\chi} \tilde{\chi} + \bar{\chi} \chi) + \left(\tilde{H}^* \tilde{H} + H^* H \right) \right].$$

With these terms, one can verify that the Lagrangian

$$\begin{aligned} \mathcal{L}_V &= \mathcal{L}_V^{(0)} + \mathcal{L}' \\ &= \text{tr} \left[-\frac{1}{4} (v_{mn})^2 - g^{\bar{z}z} v_{m\bar{z}} v_{mz} + \frac{1}{2} (D_m \sigma)^2 + g^{\bar{z}z} D_z \sigma D_{\bar{z}} \sigma - \frac{1}{2} D^2 + g^{\bar{z}z} v_{\bar{z}z} D \right. \\ &\quad - g^{\bar{z}z} \bar{F}_{\bar{z}} F_z + 2i \bar{\lambda} \gamma^m D_m \lambda - i g^{\bar{z}z} \bar{\psi}_{\bar{z}} \gamma^m D_m \psi_z - 2g^{\bar{z}z} \bar{\psi}_{\bar{z}} D_z \lambda - 2g^{\bar{z}z} D_{\bar{z}} \bar{\lambda} \psi_z \\ &\quad - g (2\bar{\lambda} [\sigma, \lambda] + g^{\bar{z}z} \bar{\psi}_{\bar{z}} [\sigma, \psi_z]) \\ &\quad \left. + \frac{1}{2} g^{\bar{z}z} \bar{\psi}_{\bar{z}} \psi_z + \bar{\lambda} \lambda + \sigma \sigma + i \sigma (D - 2g^{\bar{z}z} v_{\bar{z}z}) + \frac{1}{2} \omega_{\text{c.s.}} \right] \end{aligned}$$

of the vector multiplet is left invariant under the supersymmetry transformation (14), and the total Lagrangian $\mathcal{L} = \mathcal{L}_V + \mathcal{L}_H^{(0)} + \mathcal{L}_{\text{int}}^{(0)} + \mathcal{L}'_H$ under (14), (15).

6 Localization

In this section, we will compute the partition function of the $\mathcal{N} = 2$ theory on the $S^3 \times \Sigma$ by using the localization method. In the next subsection, we will calculate the contribution from the vector multiplet to the partition function, which can be regarded as the extension of our previous results [6] about the $\mathcal{N} = 1$ theory on $S^3 \times \mathbb{R}^2$ for the $S^3 \times \Sigma$. And we will see that the $\mathcal{N} = 1$ theory on the $S^3 \times \Sigma$ yields the partition function of two-dimensional Yang-Mills theory on Σ .

In subsection 6.2, we will proceed to the calculations of the contribution from the hypermultiplet and will find that it yields no contributions to the partition function of the $\mathcal{N} = 2$ theory on the $S^3 \times \Sigma$.

6.1 The Contribution from the Vector Multiplet

In the Lagrangian \mathcal{L}_V , the kinetic terms of the bosonic fields σ , D , F_z , and $\bar{F}_{\bar{z}}$ have the wrong sign. In order to make the path-integral well-defined, they need to be analytically continued. Therefore, we will regard the auxiliary field D as a real field and will replace the scalar field σ by $i\sigma$, where the latter σ takes the real value⁸. In addition, $\bar{F}_{\bar{z}} = (F_z)^*$.

To carry out the localization procedure, we will define the BRST transformation by setting $\bar{\epsilon}$ to zero in the supersymmetric transformation (14) and by replacing the Grass-

⁸In the previous paper [6], after the analytic continuation, σ was regarded as taking pure imaginary values.

mann odd parameter ϵ by a Grassmann even one. It yields

$$\begin{aligned}
\delta_Q v_m &= -i\bar{\lambda}\gamma_m\epsilon, & \delta_Q v_z &= 0, & \delta_Q v_{\bar{z}} &= \bar{\psi}_z\epsilon, & \delta_Q \sigma &= \bar{\lambda}\epsilon, \\
\delta_Q \lambda &= -\frac{1}{2} \left[\frac{1}{2} v_{mn}\gamma^{mn} + i\gamma^m D_m \sigma + D \right] \epsilon, & \delta_Q \bar{\lambda} &= 0, \\
\delta_Q \psi_z &= [iv_{mz}\gamma^m + D_z \sigma] \epsilon, & \delta_Q \bar{\psi}_{\bar{z}} &= \bar{F}_{\bar{z}} \epsilon^T C_3, \\
\delta_Q D &= \left[-iD_m \bar{\lambda}\gamma^m \epsilon + ig [\sigma, \bar{\lambda}] \epsilon - \frac{1}{2} \bar{\lambda}\epsilon \right], \\
\delta_Q F_z &= \epsilon^T C_3 \left[-i\gamma^m D_m \psi_z - 2D_z \lambda - ig [\sigma, \psi_z] + \frac{1}{2} \psi_z \right], & \delta_Q \bar{F}_{\bar{z}} &= 0,
\end{aligned} \tag{16}$$

which is in fact nilpotent; $\delta_Q^2 = 0$. Using the BRST transformation (16), we will modify the Lagrangian \mathcal{L}_V into $\mathcal{L}_V - t\mathcal{L}_{VQ}$ with a parameter t , where

$$\mathcal{L}_{VQ} = \delta_Q \text{tr} \left[(\delta_Q \lambda)^\dagger \lambda + \frac{1}{2} g^{\bar{z}z} (\delta_Q \psi_z)^\dagger \psi_z + \frac{1}{2} g^{\bar{z}z} \bar{\psi}_{\bar{z}} (\delta_Q \bar{\psi}_{\bar{z}})^\dagger \right]. \tag{17}$$

The bosonic part of the extra Lagrangian \mathcal{L}_{VQ} gives

$$\begin{aligned}
\mathcal{L}_{VQ}^{(B)} &= \frac{1}{2} \text{tr} \left[\frac{1}{4} v^{mn} v_{mn} + g^{\bar{z}z} g^{mn} v_{m\bar{z}} v_{nz} + \frac{1}{2} D^m \sigma D_m \sigma + g^{\bar{z}z} D_{\bar{z}} \sigma D_z \sigma + \frac{1}{2} D^2 \right. \\
&\quad \left. + g^{\bar{z}z} \bar{F}_{\bar{z}} F_z + ik^m g^{\bar{z}z} (v_{mz} D_{\bar{z}} \sigma - v_{m\bar{z}} D_z \sigma) + ik_m \epsilon^{mnk} g^{\bar{z}z} v_{n\bar{z}} v_{kz} \right]
\end{aligned}$$

where the Killing vector k_m was defined by

$$k_m = \bar{\epsilon} \gamma_m \epsilon$$

with the normalization $(\bar{\epsilon}\epsilon) = 1$. On the other hand, the fermionic part of \mathcal{L}_{VQ} gives

$$\begin{aligned}
\mathcal{L}_{VQ}^{(F)} &= i \text{tr} \left[-\bar{\lambda} \gamma^m D_m \lambda - \frac{i}{2} \bar{\lambda} \lambda + g \bar{\lambda} [\sigma, \lambda] + \frac{1}{2} g^{\bar{z}z} \bar{\psi}_{\bar{z}} \gamma^m D_m \psi_z - \frac{i}{4} g^{\bar{z}z} \bar{\psi}_{\bar{z}} \psi_z \right. \\
&\quad \left. + \frac{1}{2} g^{\bar{z}z} (g \bar{\psi}_{\bar{z}} [\sigma, \psi_z] - ik_m \bar{\psi}_{\bar{z}} \gamma^m \psi_z + 2i\bar{\lambda} D_{\bar{z}} \psi_z - i\bar{\psi}_{\bar{z}} D_z \lambda + ik_m \bar{\psi}_{\bar{z}} \gamma^m D_z \lambda) \right].
\end{aligned}$$

In the large t limit, $t \rightarrow \infty$, the fixed point, which is a solution to

$$\left[\frac{1}{2} v_{mn} \gamma^{mn} + i\gamma^m D_m \sigma + D \right] \epsilon = 0, \quad [v_{mz} \gamma^m - iD_z \sigma] \epsilon = 0, \quad F_z = 0,$$

gives the dominant contribution to the partition function. In fact, the fixed point is given [21] by

$$v_m = 0, \quad D = 0, \quad F = 0, \quad v_z = v_z(z, \bar{z}), \quad \sigma = \sigma(z, \bar{z}), \quad D_z \sigma = 0. \tag{18}$$

Substituting the background (18) into the Lagrangian \mathcal{L}_V , one finds that the additional Lagrangian \mathcal{L}'_V only contributes and yields

$$\mathcal{L}_{YM} = \text{tr}[-\sigma\sigma + 2\sigma g^{\bar{z}z} v_{\bar{z}z}], \tag{19}$$

which would give the action of the two-dimensional Yang-Mills theory if one could integrate out the scalar field σ . However, in this case, the scalar field σ must obey the condition $D_z\sigma = 0$, and it isn't allowed to perform the Gaussian integration over the whole functional space of the σ .

Around the fixed points, we will perform the path integral over the quantum fluctuations. Since the bosonic fields σ , v_z , and $v_{\bar{z}}$ have a non-trivial background as the fixed point, we will expand the fields as

$$\sigma = \sigma(z, \bar{z}) + \frac{1}{\sqrt{t}}\tilde{\sigma}(x^m, z, \bar{z}), \quad v_z = v_z(z, \bar{z}) + \frac{1}{\sqrt{t}}\tilde{v}_z(x^m, z, \bar{z}),$$

while the other fields are rescaled as $\Phi \rightarrow (1/\sqrt{t})\tilde{\Phi}$.

One also needs the gauge-fixing procedure for the computation of the path integral. We will follow [21] and add to \mathcal{L}_{VQ} the gauge-fixing term and the ghost term

$$\text{tr}[\bar{C}\nabla_m D^m C + B\nabla^m v_m]. \quad (20)$$

There remains the residual gauge symmetry, under which

$$\sigma \rightarrow \sigma + ig[\omega(z, \bar{z}), \sigma], \quad v_z \rightarrow v_z - D_z\omega(z, \bar{z}), \quad (21)$$

where the gauge transformation parameter ω is constant along the S^3 . The symmetry is the redundancy of the backgrounds σ and v_z , but not of the fluctuations, and the gauge fixing procedure can be carried out in a similar way to the two-dimensional Yang-Mills theory, as explained in [19, 20]. Following [19, 20], one can make use of the residual symmetry (21) to put the background $\sigma(z, \bar{z})$ in the Cartan subalgebra of the Lie algebra of G such that

$$\sigma(z, \bar{z}) = \sum_{i=1}^r \sigma_i H_i. \quad (22)$$

Recall that H_i ($i = 1, \dots, r$) are the generators of the Cartan subalgebra of G of rank r , and the localization condition $D_z\sigma = 0$ in (18) implies that the background $v_z(z, \bar{z})$ should also be in the Cartan subalgebra as

$$v_z(z, \bar{z}) = \sum_{i=1}^r v_z^i(z, \bar{z}) H_i, \quad (23)$$

and furthermore that σ_i ($i = 1, \dots, r$) are constant with respect to the local coordinates z, \bar{z} of the Σ .

Therefore, to impose the gauge-fixing condition (22), we will follow the same BRST quantization procedure as for the two-dimensional Yang-Mills theory in [19, 20]. Introducing another auxiliary field

$$b(z, \bar{z}) = \sum_{\alpha \in \Lambda} b^\alpha(z, \bar{z}) E_\alpha,$$

and another pair of ghost fields

$$c(z, \bar{z}) = \sum_{\alpha \in \Lambda} c^\alpha(z, \bar{z}) E_\alpha, \quad \bar{c}(z, \bar{z}) = \sum_{\alpha \in \Lambda} \bar{c}^\alpha(z, \bar{z}) E_\alpha,$$

where Λ is the set of all the root⁹ of the Lie algebra of G , and the root generators E_α satisfy the algebra

$$[H_i, E_\alpha] = \alpha_i E_\alpha, \quad [E_\alpha, E_{-\alpha}] = \sum_{i=1}^r \alpha_i H_i \equiv \alpha \cdot H,$$

we will add another gauge-fixing term and the ghost term

$$\sum_{\alpha \in \Lambda} [ib^{-\alpha} \sigma^\alpha - ig(\alpha \cdot \sigma) \bar{c}^{-\alpha} c^\alpha],$$

where $(\alpha \cdot \sigma) = \sum_{i=1}^r \alpha_i \sigma^i$. After the integration over the root part σ^α , the auxiliary field b^α and the ghosts \bar{c}^α, c^α , the path-integral measure of the scalar field $\sigma(z, \bar{z})$ results in the finite-dimensional integral over σ_i ($i = 1, \dots, r$), the ghosts give the one-loop determinant

$$\prod_{\alpha \in \Lambda} \text{Det}_{(0,0)} [ig(\alpha \cdot \sigma)], \quad (24)$$

which is the same contribution as the ghosts do in the two-dimensional Yang-Mills theory.

Now, let us proceed to the calculations of the one-loop determinants of the vector multiplet. To this end, we will follow the same procedure as in [21, 22], - expanding all the fields in terms of the spherical harmonics on S^3 and performing the Gaussian integration over them.

On the unit round S^3 , we will use the vielbein $e^a = e^a_m dx^m$ obeying $de^a = \epsilon^{abc} e^b \wedge e^c$ and the spin connection $\omega^{ab} = \epsilon^{abc} e^c$. We define the three-dimensional Hodge duality as

$$*e^a = \frac{1}{2} \epsilon^{abc} e^b \wedge e^c, \quad *(e^a \wedge e^b) = \epsilon^{abc} e^c,$$

with $*1$ the volume form, and two operators ι_k and S^a acting on the vielbein by

$$\iota_k e^a = e^a_m k^m = k^a, \quad S^a e^b = i\epsilon^{abc} e^c,$$

with $k^m = \bar{\epsilon} \gamma^m \epsilon$ is the Killing vector field.

Expanding the Lagrangian \mathcal{L}_{VQ} in terms of the fluctuations up to quadratic order, one finds that the bosonic part $\mathcal{L}_{VQ}^{(B)}$ in a differential form notation gives

$$\begin{aligned} & \frac{1}{2} \text{tr} \left[\frac{1}{2} d\tilde{v} \wedge *d\tilde{v} + \frac{1}{2} D\tilde{\sigma} \wedge *D\tilde{\sigma} + g^{\bar{z}z} (d\tilde{v}_{\bar{z}} - D_{\bar{z}}\tilde{v}) \wedge * (d\tilde{v}_z - D_z\tilde{v}) \right. \\ & + g^{\bar{z}z} (D_{\bar{z}}\tilde{\sigma} - ig[\sigma, \tilde{v}_{\bar{z}}]) (D_z\tilde{\sigma} - ig[\sigma, \tilde{v}_z]) * 1 + ig^{\bar{z}z} (D_{\bar{z}}\tilde{\sigma} - ig[\sigma, \tilde{v}_{\bar{z}}]) \iota_k (d\tilde{v}_z - D_z\tilde{v}) * 1 \\ & \left. - ig^{\bar{z}z} (D_z\tilde{\sigma} - ig[\sigma, \tilde{v}_z]) \iota_k (d\tilde{v}_{\bar{z}} - D_{\bar{z}}\tilde{v}) * 1 - g^{\bar{z}z} (d\tilde{v}_{\bar{z}} - D_{\bar{z}}\tilde{v}) \wedge * [(k \cdot S) (d\tilde{v}_z - D_z\tilde{v})] \right], \end{aligned} \quad (25)$$

with $(k \cdot S) = k^a S^a$, where the form notation denotes

$$\tilde{v} = \tilde{v}_m dx^m, \quad D\tilde{\sigma} = d\tilde{\sigma} - ig[\sigma, \tilde{v}] = (\partial_m \tilde{\sigma} - ig[\sigma, \tilde{v}_m]) dx^m.$$

Note that the gauge fields in the covariant derivatives D_z and $D_{\bar{z}}$ here are the background v_z and $v_{\bar{z}}$, respectively.

⁹Here, α denotes a root of the Lie algebra of G , but not the index of the $SU(2)$ symmetry. We hope which one we mean will be clear from the context.

For brevity, we will omit the tilde \sim for the fermionic fluctuations, and then the fermionic part $\mathcal{L}_{VQ}^{(F)}$ gives

$$\begin{aligned} \text{tr} \left[-i\bar{\lambda}\gamma^m\nabla_m\lambda + \frac{1}{2}\bar{\lambda}\lambda + ig\bar{\lambda}[\sigma, \lambda] + \frac{i}{2}g^{\bar{z}z}\bar{\psi}_{\bar{z}}\gamma^m\nabla_m\psi_z + \frac{1}{4}g^{\bar{z}z}\bar{\psi}_{\bar{z}}\psi_z \right. \\ \left. + \frac{i}{2}gg^{\bar{z}z}\bar{\psi}_{\bar{z}}[\sigma, \psi_z] + \frac{1}{2}g^{\bar{z}z}k_m\bar{\psi}_{\bar{z}}\gamma^m\psi_z + g^{\bar{z}z}D_{\bar{z}}\bar{\lambda}\cdot\psi_z + \frac{1}{2}g^{\bar{z}z}\bar{\psi}_{\bar{z}}(1-k_m\gamma^m)D_z\lambda \right], \end{aligned} \quad (26)$$

with the covariant derivatives D_z and $D_{\bar{z}}$ including only the background gauge fields v_z and $v_{\bar{z}}$, respectively.

In terms of the scalar spherical harmonics $\varphi_{l,m,\tilde{m}}$ ($l = 0, 1/2, 1, 3/2, \dots; -l \leq m \leq l; -l \leq \tilde{m} \leq l$), with the properties

$$\begin{aligned} d^\dagger d\varphi_{l,m,\tilde{m}} &= - * d * d\varphi_{l,m,\tilde{m}} = 4l(l+1)\varphi_{l,m,\tilde{m}}, \\ (\varphi_{l,m,\tilde{m}})^* &= \varphi_{l,-m,-\tilde{m}}, \\ \int_{S^3} (\varphi_{l',m',\tilde{m}'}^*)^* \varphi_{l,m,\tilde{m}} * 1 &= \delta_{l'l}\delta_{m,m'}\delta_{\tilde{m},\tilde{m}'}, \end{aligned}$$

on the S^3 , the fields $\tilde{\sigma}$, \tilde{v}_z are expanded as

$$\tilde{\sigma} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\tilde{m}=-l}^l \tilde{\sigma}_{l,m,\tilde{m}}(z, \bar{z})\varphi_{l,m,\tilde{m}}(x), \quad \tilde{v}_z = \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\tilde{m}=-l}^l \tilde{v}_{z,l,m,\tilde{m}}(z, \bar{z})\varphi_{l,m,\tilde{m}}(x).$$

The vector spherical harmonics on the S^3 are combined by the scalar spherical harmonics $\varphi_{l,m,\tilde{m}}$ and the vielbein e^a . In particular, we will take the vielbein as the eigenstate of the operators $S^a S^a$ and S^3 as

$$e^{\pm 1} = \mp \frac{1}{\sqrt{2}} (e^1 \pm ie^2), \quad e^0 = e^3,$$

and form the vector spherical harmonics on the S^3 ,

$$E_{J,M;l,\tilde{m}} = \sum_{m=-l}^l \sum_{s=-1}^{+1} \langle l, m; 1, s | J, M \rangle \varphi_{l,m,\tilde{m}} e^s,$$

where $\langle l, m; 1, s | J, M \rangle$ are the Clebsch-Gordan coefficients of the spin l representation and the spin 1 representation of the $SU(2)$ group into the spin J representation, with $J = (l-1), l, (l+1)$. The Clebsch-Gordan coefficients are listed in appendix B. They have the properties

$$\begin{aligned} *d E_{l+1,M;l,\tilde{m}} &= 2(l+1)E_{l+1,M;l,\tilde{m}}, & d * E_{l+1,M;l,\tilde{m}} &= 0, & (l = 0, \frac{1}{2}, 1, \dots) \\ *d E_{l-1,M;l,\tilde{m}} &= -2l E_{l-1,M;l,\tilde{m}}, & d * E_{l-1,M;l,\tilde{m}} &= 0, & (l = 1, \frac{3}{2}, 2, \dots) \\ *d E_{l,M;l,\tilde{m}} &= 0, & E_{l,M;l,\tilde{m}} &= -\frac{i}{2} \sqrt{\frac{1}{l(l+1)}} d\varphi_{l,m,\tilde{m}}, & (l = \frac{1}{2}, 1, \frac{3}{2}, \dots) \end{aligned}$$

and form the orthonormal basis

$$\int_{S^3} (E_{J',M';l',\tilde{m}'}^*)^* \wedge *E_{J,M;l,\tilde{m}} = \delta_{J,J'}\delta_{M,M'}\delta_{l,l'}\delta_{m,m'}.$$

In terms of them, the gauge field \tilde{v}_m is expanded as $\tilde{v} = \tilde{v}_+ + \tilde{v}_- + \tilde{v}_L$, where

$$\begin{aligned}\tilde{v}_+ &= \sum_{l=0}^{\infty} \sum_{M=-l-1}^{l+1} \sum_{\tilde{m}=-l}^l \tilde{v}_{l+1,M;l,\tilde{m}}(z, \bar{z}) E_{l+1,M;l\tilde{m}}(x), \\ \tilde{v}_- &= \sum_{l=1}^{\infty} \sum_{M=-(l-1)}^{l-1} \sum_{\tilde{m}=-l}^l \tilde{v}_{l-1,M;l,\tilde{m}}(z, \bar{z}) E_{l-1,M;l\tilde{m}}(x), \\ \tilde{v}_L &= \sum_{l=1/2}^{\infty} \sum_{M=-l}^l \sum_{\tilde{m}=-l}^l \tilde{v}_{l,M;l,\tilde{m}}(z, \bar{z}) E_{l,M;l\tilde{m}}(x) \\ &= -\frac{i}{2}d \sum_{l=1/2}^{\infty} \sum_{M=-l}^l \sum_{\tilde{m}=-l}^l \sqrt{\frac{1}{l(l+1)}} \tilde{v}_{l,M;l,\tilde{m}}(z, \bar{z}) E_{l,M;l\tilde{m}}(x) = d\tilde{u}_L,\end{aligned}$$

with \tilde{v}_{\pm} the transverse modes and \tilde{v}_L the longitudinal mode.

Substituting these expansions into the bosonic part (25), one sees that the longitudinal mode \tilde{v}_L can be eliminated in $\mathcal{L}_{VQ}^{(B)}$ by shifting the fields \tilde{v}_z and $\tilde{\sigma}$ as

$$\tilde{v}_z \rightarrow \tilde{v}_z + D_z \tilde{u}_L, \quad \tilde{\sigma} \rightarrow \tilde{\sigma} + ig [\sigma, \tilde{u}_L].$$

Therefore, the longitudinal mode \tilde{v}_L appears only in the gauge fixing term (20), which up to quadratic order yields

$$\bar{C}d * dC + Bd * \tilde{v} = \bar{C}d * dC + Bd * d\tilde{u}_L.$$

It is obvious that the one-loop determinant from B and \tilde{u}_L exactly cancels the one-loop determinant from the ghosts \bar{C} and C .

As for the operators ι_k and $(k \cdot S)$ with the Killing vector k_a appearing in (25), one will take the Killing spinor ϵ as constant, and then the Killing vector $k^a = \bar{\epsilon}\gamma^a\epsilon$ is also constant. Since $k^a k_a = 1$ with the normalization $\bar{\epsilon}\epsilon = 1$, we will choose it as $k^a = \delta^a_3$, as in [21]. Therefore, one obtains the formulas

$$\begin{aligned}\int_{S^3} (\varphi_{l',m',\tilde{m}'}^*)^* \iota_k E_{J,M;l,\tilde{m}} * 1 &= \int_{S^3} (\varphi_{l',m',\tilde{m}'}^*)^* e^3 \wedge * E_{J,M;l,\tilde{m}} = \delta_{l,l'} \delta_{\tilde{m},\tilde{m}'} \langle l, m'; 1, s=0 | J, M \rangle, \\ \int_{S^3} (E_{J',M';l',\tilde{m}'}) \wedge * [(k \cdot S) E_{J,M;l,\tilde{m}}] &= \delta_{l,l'} \delta_{\tilde{m},\tilde{m}'} \langle \langle J', M' | S^3 | J, M \rangle \rangle.\end{aligned}$$

For the coefficients $\langle l, m'; 1, s=0 | J, M \rangle$, $\langle \langle J', M' | S^3 | J, M \rangle \rangle$, see the list in appendix B.

We are not interested in the overall constant of the partition function, but in its dependence on the background σ^i and v_z^i . The Cartan part $\tilde{\Phi}^i$ of the fluctuations and the root part $\tilde{\Phi}^\alpha$ of them are completely decoupled from each other, and the Cartan part doesn't yield the contributions which has the dependence of the background. We will thus focus on the contributions from the root part of the fluctuations, but one can easily see that the contributions from the Cartan part of the fluctuations can be obtained by setting $(\alpha \cdot \sigma)$ to zero and by replacing $\alpha \in \Lambda$ by i running from 1 to r in the results of the contributions from the root part.

In the action $S_{VQ}^{(B)} = \int_{S^3} \mathcal{L}_{VQ}^{(B)}$ given in terms of the modes of the fluctuations, after completing the square by shifting the variables, one finds that

$$S_{VQ}^{(B)} = \sum_{\alpha \in \Lambda_+} \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\tilde{m}=-l}^l S_{VQ;\alpha,l,m,\tilde{m}}^{(B)} + S_{VQ;H}^{(B)},$$

with Λ_+ the set of the positive roots, where $S_{VQ;H}^{(B)}$ consists of the modes in the Cartan subalgebra of G . The modes in each $S_{VQ;\alpha,l,m,\tilde{m}}^{(B)}$ decouple from the modes in the rest of $S_{VQ}^{(B)}$. The action $S_{VQ;\alpha,l,m,\tilde{m}}^{(B)}$ of the modes for $l \geq 1$, $-(l-1) \leq m \leq l-1$, $-l \leq \tilde{m} \leq l$, is given by

$$S_{VQ;\alpha,l,m,\tilde{m}}^{(B)} = \frac{1}{2} \left[K_{l,m}^\alpha g^{\bar{z}z} |\tilde{v}_{z,l,m,\tilde{m}}^\alpha|^2 + K_{l,-m}^\alpha g^{\bar{z}z} |\tilde{v}_{\bar{z},l,m,\tilde{m}}^\alpha|^2 + (\tilde{\sigma}_{l,m,\tilde{m}}^\alpha)^* \Delta_{l,m}^\alpha \tilde{\sigma}_{l,m,\tilde{m}}^\alpha \right. \\ \left. + \begin{pmatrix} (\tilde{v}_{l+1,m;l,\tilde{m}}^\alpha)^* & (\tilde{v}_{l-1,m;l,\tilde{m}}^\alpha)^* \end{pmatrix} \begin{pmatrix} A_{l,m}^\alpha & B_{l,m}^\alpha \\ C_{l,m}^\alpha & D_{l,m}^\alpha \end{pmatrix} \begin{pmatrix} \tilde{v}_{l+1,m;l,\tilde{m}}^\alpha \\ \tilde{v}_{l-1,m;l,\tilde{m}}^\alpha \end{pmatrix} \right],$$

where the above operators are defined by

$$K_{l,m}^\alpha = [4l(l+1) - 4m + g^2(\alpha \cdot \sigma)^2], \\ \Delta_{l,m}^\alpha = -\frac{4(l-m)(l+m+1)}{K_{l,m}^\alpha} g^{\bar{z}z} D_{\bar{z}} D_z - \frac{4(l+m)(l-m+1)}{K_{l,-m}^\alpha} g^{\bar{z}z} D_z D_{\bar{z}} + 4l(l+1), \\ A_{l,m}^\alpha = U_{l,m}^\alpha \frac{1}{\Delta_{l,m}^\alpha} V_{l,m}^\alpha + u_{l,m} v_{l,m}, \quad B_{l,m}^\alpha = U_{l,m}^\alpha \frac{1}{\Delta_{l,m}^\alpha} \tilde{V}_{l,m}^\alpha + u_{l,m} \tilde{v}_{l,m}, \\ C_{l,m}^\alpha = \tilde{U}_{l,m}^\alpha \frac{1}{\Delta_{l,m}^\alpha} V_{l,m}^\alpha + \tilde{u}_{l,m} v_{l,m}, \quad D_{l,m}^\alpha = \tilde{U}_{l,m}^\alpha \frac{1}{\Delta_{l,m}^\alpha} \tilde{V}_{l,m}^\alpha + \tilde{u}_{l,m} \tilde{v}_{l,m},$$

with

$$U_{l,m}^\alpha = 2(l+m) [2(l+1) + ig(\alpha \cdot \sigma)] \sqrt{\frac{(l+1)(l+m+1)}{(2l+1)(l-m+1)}} \left[\frac{2}{K_{l,-m}^\alpha} g^{\bar{z}z} D_z D_{\bar{z}} - \frac{l}{l+m} \right], \\ \tilde{U}_{l,m}^\alpha = 2(l+m+1) [2l - ig(\alpha \cdot \sigma)] \sqrt{\frac{l(l+m)}{(2l+1)(l-m)}} \left[\frac{2}{K_{l,-m}^\alpha} g^{\bar{z}z} D_z D_{\bar{z}} - \frac{l+1}{l-m+1} \right], \\ V_{l,m}^\alpha = 2(l-m) [2(l+1) - ig(\alpha \cdot \sigma)] \sqrt{\frac{(l+1)(l-m+1)}{(2l+1)(l+m+1)}} \left[\frac{2}{K_{l,m}^\alpha} g^{\bar{z}z} D_{\bar{z}} D_z - \frac{l}{l-m} \right], \\ \tilde{V}_{l,m}^\alpha = 2(l-m+1) [2l + ig(\alpha \cdot \sigma)] \sqrt{\frac{l(l-m)}{(2l+1)(l+m)}} \left[\frac{2}{K_{l,m}^\alpha} g^{\bar{z}z} D_{\bar{z}} D_z - \frac{l+1}{l+m+1} \right], \\ u_{l,m} = \sqrt{\frac{(l+1)(l-m+1)}{(2l+1)(l+m+1)}} [2(l+1) + ig(\alpha \cdot \sigma)], \\ v_{l,m} = \sqrt{\frac{(l+1)(l+m+1)}{(2l+1)(l-m+1)}} [2(l+1) - ig(\alpha \cdot \sigma)], \\ \tilde{u}_{l,m} = -\sqrt{\frac{l(l+m)}{(2l+1)(l-m)}} [2l - ig(\alpha \cdot \sigma)], \quad \tilde{v}_{l,m} = -\sqrt{\frac{l(l-m)}{(2l+1)(l+m)}} [2l + ig(\alpha \cdot \sigma)].$$

Note here that the covariant derivatives $D_{\bar{z}}$, D_z acting on the root part of a field, Φ^α gives

$$D_{\bar{z}} \Phi^\alpha = \partial_{\bar{z}} \Phi^\alpha + ig \sum_{i=1}^r \alpha_i v_{\bar{z}}^i \Phi^\alpha, \quad D_z \Phi^\alpha = \partial_z \Phi^\alpha + ig \sum_{i=1}^r \alpha_i v_z^i \Phi^\alpha.$$

One can now see that the one-loop determinant from these modes yields

$$\begin{aligned}
& \prod_{\alpha \in \Lambda_+} \prod_{l=1}^{\infty} \prod_{m=-(l-1)}^{l-1} \prod_{\tilde{m}=-l}^l \left[\frac{1}{\text{Det}_{(1,0)} [K_{l,m}^{\alpha}] \text{Det}_{(0,1)} [K_{l,-m}^{\alpha}] \text{Det}_{(0,0)} [\Delta_{l,m}^{\alpha}]} \right. \\
& \quad \left. \times \frac{\text{Det}_{(0,0)} [\Delta_{l,m}^{\alpha}]}{\text{Det}_{(0,0)} [\tilde{u}_{l,m} U_{l,m}^{\alpha} - u_{l,m} \tilde{U}_{l,m}^{\alpha}] \text{Det}_{(0,0)} [\tilde{v}_{l,m} V_{l,m}^{\alpha} - v_{l,m} \tilde{V}_{l,m}^{\alpha}]} \right] \\
& = \prod_{\alpha \in \Lambda_+} \prod_{l=1}^{\infty} \prod_{m=-(l-1)}^{l-1} \prod_{\tilde{m}=-l}^l \left[\frac{1}{\text{Det}_{(0,0)} [4l(l+1)]} \frac{\text{Det}_{(0,0)} [K_{l,m}^{\alpha}] \text{Det}_{(0,0)} [K_{l,-m}^{\alpha}]}{\text{Det}_{(1,0)} [K_{l,m}^{\alpha}] \text{Det}_{(0,1)} [K_{l,-m}^{\alpha}]} \right. \\
& \quad \left. \times \frac{\text{Det}_{(0,0)} [4l^2 + g^2(\alpha \cdot \sigma)^2] \text{Det}_{(0,0)} [4(l+1)^2 + g^2(\alpha \cdot \sigma)^2]}{\text{Det}_{(0,0)} [2g^{\bar{z}z} D_{\bar{z}} D_z - K_{l,m}^{\alpha}] \text{Det}_{(0,0)} [2g^{\bar{z}z} D_z D_{\bar{z}} - K_{l,-m}^{\alpha}]} \right] \\
& = \prod_{\alpha \in \Lambda} \prod_{l=1}^{\infty} \prod_{m=-(l-1)}^{l-1} \prod_{\tilde{m}=-l}^l \left[\frac{\text{Det}_{(0,0)} [K_{l,m}^{\alpha}]}{\text{Det}_{(1,0)} [K_{l,m}^{\alpha}]} \frac{1}{\text{Det}_{(0,0)} [2g^{\bar{z}z} D_{\bar{z}} D_z - K_{l,m}^{\alpha}]} \right. \\
& \quad \left. \times \frac{\text{Det}_{(0,0)} [2(l+1) + ig(\alpha \cdot \sigma)] \text{Det}_{(0,0)} [-2l + ig(\alpha \cdot \sigma)]}{\sqrt{\text{Det}_{(0,0)} [4l(l+1)]}} \right].
\end{aligned}$$

Further, after some similar algebra, the action $S_{VQ;\alpha,l,m,\tilde{m}}^{(B)}$ of the modes for $l \geq 1/2$, $m = -l$, $-l \leq \tilde{m} \leq l$ can be read as

$$\begin{aligned}
& \frac{1}{2} \left[K_{l,-l}^{\alpha} g^{\bar{z}z} |\tilde{v}_{z,l,-l,\tilde{m}}^{\alpha}|^2 + K_{l,l}^{\alpha} g^{\bar{z}z} |\tilde{v}_{\bar{z},l,-l,\tilde{m}}^{\alpha}|^2 + \frac{4l}{K_{l,-l}^{\alpha}} (\tilde{\sigma}_{l,-l,\tilde{m}}^{\alpha})^* [-2g^{\bar{z}z} D_{\bar{z}} D_z + (l+1)K_{l,-l}^{\alpha}] \tilde{\sigma}_{l,-l,\tilde{m}}^{\alpha} \right. \\
& \quad \left. + (l+1) (\tilde{v}_{l+1,-l;l,\tilde{m}}^{\alpha})^* \frac{K_{l,-l-1}^{\alpha}}{-2g^{\bar{z}z} D_{\bar{z}} D_z + (l+1)K_{l,-l}^{\alpha}} [-2g^{\bar{z}z} D_{\bar{z}} D_z + K_{l,-l}^{\alpha}] \tilde{v}_{l+1,-l;l,\tilde{m}}^{\alpha} \right],
\end{aligned}$$

and the one for $l \geq 1/2$, $m = +l$, $-l \leq \tilde{m} \leq l$ as

$$\begin{aligned}
& \frac{1}{2} \left[K_{l,l}^{\alpha} g^{\bar{z}z} |\tilde{v}_{z,l,l,\tilde{m}}^{\alpha}|^2 + K_{l,-l}^{\alpha} g^{\bar{z}z} |\tilde{v}_{\bar{z},l,l,\tilde{m}}^{\alpha}|^2 + \frac{4l}{K_{l,-l}^{\alpha}} (\tilde{\sigma}_{l,l,\tilde{m}}^{\alpha})^* [-2g^{\bar{z}z} D_z D_{\bar{z}} + (l+1)K_{l,-l}^{\alpha}] \tilde{\sigma}_{l,l,\tilde{m}}^{\alpha} \right. \\
& \quad \left. + (l+1) (\tilde{v}_{l+1,l;l,\tilde{m}}^{\alpha})^* \frac{K_{l,-l-1}^{\alpha}}{-2g^{\bar{z}z} D_z D_{\bar{z}} + (l+1)K_{l,-l}^{\alpha}} [-2g^{\bar{z}z} D_z D_{\bar{z}} + K_{l,-l}^{\alpha}] \tilde{v}_{l+1,l;l,\tilde{m}}^{\alpha} \right].
\end{aligned}$$

These sectors with $l \geq 1/2$, $m = \pm l$, $-l \leq \tilde{m} \leq l$ gives the one-loop determinant

$$\begin{aligned}
& \prod_{\alpha \in \Lambda_+} \prod_{l=1/2}^{\infty} \prod_{\tilde{m}=-l}^l \frac{1}{\text{Det}_{(0,0)} [l^2(l+1)^2]} \left(\frac{\text{Det}_{(0,0)} [K_{l,-l}^{\alpha}]}{\text{Det}_{(1,0)} [K_{l,l}^{\alpha}] \text{Det}_{(0,1)} [K_{l,-l}^{\alpha}] \text{Det}_{(0,0)} [K_{l,-l-1}^{\alpha}]} \right)^2 \\
& \quad \times \frac{1}{\text{Det}_{(0,0)} [-2g^{\bar{z}z} D_z D_{\bar{z}} + K_{l,-l}^{\alpha}] \text{Det}_{(0,0)} [-2g^{\bar{z}z} D_{\bar{z}} D_z + K_{l,-l}^{\alpha}]} \\
& = \prod_{\alpha \in \Lambda} \prod_{l=1/2}^{\infty} \prod_{\tilde{m}=-l}^l \frac{1}{\text{Det}_{(0,0)} [l(l+1)]} \frac{\text{Det}_{(0,0)} [K_{l,-l}^{\alpha}]}{\text{Det}_{(1,0)} [K_{l,l}^{\alpha}] \text{Det}_{(0,1)} [K_{l,-l}^{\alpha}] \text{Det}_{(0,0)} [K_{l,-l-1}^{\alpha}]} \\
& \quad \times \frac{1}{\text{Det}_{(0,0)} [-2g^{\bar{z}z} D_{\bar{z}} D_z + K_{l,-l}^{\alpha}]}.
\end{aligned}$$

Since the actions $S_{VQ;\alpha,l,\pm(l+1),\tilde{m}}^{(B)}$ of the modes with $l \geq 0$ take simple forms, we will give the sum

$$\begin{aligned} & S_{VQ;\alpha,l,l+1,\tilde{m}}^{(B)} + S_{VQ;\alpha,l,-(l+1),\tilde{m}}^{(B)} \\ &= \frac{1}{2} (\tilde{v}_{l+1,l+1;l,\tilde{m}}^\alpha)^* [-2g^{\bar{z}z} D_z D_{\bar{z}} + K_{l,-l-1}^\alpha] \tilde{v}_{l+1,l+1;l,\tilde{m}}^\alpha \\ & \quad + \frac{1}{2} (\tilde{v}_{l+1,-(l+1);l,\tilde{m}}^\alpha)^* [-2g^{\bar{z}z} D_{\bar{z}} D_z + K_{l,-l-1}^\alpha] \tilde{v}_{l+1,-(l+1);l,\tilde{m}}^\alpha, \end{aligned}$$

to yield the one-loop determinant

$$\prod_{\alpha \in \Lambda_+} \prod_{l=0}^{\infty} \prod_{\tilde{m}=-l}^l \frac{1}{\text{Det}_{(0,0)} [-2g^{\bar{z}z} D_z D_{\bar{z}} + K_{l,-l-1}^\alpha] \text{Det}_{(0,0)} [-2g^{\bar{z}z} D_{\bar{z}} D_z + K_{l,-l-1}^\alpha]}.$$

Finally, one can find the action $S_{VQ;\alpha,0,0,0}^{(B)}$

$$\frac{1}{2} \left[K_{0,-1}^\alpha |\tilde{v}_{1,0;0,0}^\alpha|^2 + g^{\bar{z}z} |D_z \tilde{v}_{1,0;0,0}^\alpha + g(\alpha \cdot \sigma) \tilde{v}_{z,0,0,0}^\alpha|^2 + g^{\bar{z}z} |D_{\bar{z}} \tilde{v}_{1,0;0,0}^\alpha - g(\alpha \cdot \sigma) \tilde{v}_{\bar{z},0,0,0}^\alpha|^2 \right],$$

of the modes with $l = 0$, and it gives the one-loop determinant

$$\prod_{\alpha \in \Lambda_+} \frac{1}{\text{Det}_{(0,0)} [K_{0,-1}^\alpha] \text{Det}_{(1,0)} [g(\alpha \cdot \sigma)] \text{Det}_{(0,1)} [g(\alpha \cdot \sigma)]}.$$

Let us turn to the one-loop determinant from the fermionic fluctuations. To this end, we will identify the spin operator \mathcal{S}^a ($a = 1, 2, 3$) with the gamma matrix $(1/2)\gamma^a$ ($a = 1, 2, 3$), respectively, and one can easily verify that they obeys the $SU(2)$ algebra

$$[\mathcal{S}^a, \mathcal{S}^b] = i\epsilon^{abc} \mathcal{S}^c.$$

One can easily see that the left-invariant vector fields $L_a = -(i/2)e_a^m \nabla_m$ ($a = 1, 2, 3$) also satisfy the $SU(2)$ algebra

$$[L_a, L_b] = i\epsilon_{abc} L_c.$$

Therefore, on the spinors $\tilde{\lambda}, \tilde{\psi}$, one finds that

$$\begin{aligned} \gamma^m \nabla_m \tilde{\lambda} &= \gamma^a \left(2iL_a + \frac{1}{4} (\omega_a)^{bc} \gamma^{bc} \right) \tilde{\lambda} = 2i [(L_a + \mathcal{S}_a)^2 - L_a L_a] \tilde{\lambda}, \\ \gamma^m \nabla_m \tilde{\psi} &= \gamma^a \left(2iL_a + \frac{1}{4} (\omega_a)^{bc} \gamma^{bc} \right) \tilde{\psi} = 2i [(L_a + \mathcal{S}_a)^2 - L_a L_a] \tilde{\psi}. \end{aligned}$$

In order to obtain the spherical harmonics expansion of the spinors $\tilde{\lambda}, \tilde{\psi}$, it is useful to introduce the eigenspinors $\eta_{J,M;l,\tilde{m}}$ of the operator $\gamma^m \nabla_m$ by

$$\eta_{J,M;l,\tilde{m}} = \sum_{m=-l}^l \sum_{s=\pm(1/2)} \langle l, m; \frac{1}{2}, s | J, M \rangle \varphi_{l,m,\tilde{m}} \zeta'_s,$$

with $\langle l, m; \frac{1}{2}, s | J, M \rangle$ the Clebsch-Gordan coefficients of the spin l representation and the spin $1/2$ representation into the spin $J = l \pm 1/2$ representation, where the spinors ζ'_\pm satisfy that $\mathcal{S}^3 \zeta'_\pm = \pm(1/2)\zeta'_\pm$.

They have their eigenvalues

$$\begin{aligned}\gamma^m \nabla_m \eta_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} &= i(2l + \frac{3}{2}) \eta_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}, \\ \gamma^m \nabla_m \eta_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} &= -i(2l + \frac{1}{2}) \eta_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}},\end{aligned}$$

and form the orthonormalized basis

$$\int_{S^3} (\eta_{J', M'; l', \tilde{m}'}^\dagger) \eta_{J, M; l, \tilde{m}} * 1 = \delta_{J, J'} \delta_{M, M'} \delta_{l, l'} \delta_{m, m'}.$$

Substituting the spherical harmonics expansion of the spinors $\tilde{\lambda}, \tilde{\psi}$,

$$\begin{aligned}\lambda &= \sum_{l=0} \sum_{m=-l-1}^l \sum_{\tilde{m}=-l}^l \lambda_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} \eta_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} + \sum_{l=1/2}^{l-1} \sum_{m=-l}^{l-1} \sum_{\tilde{m}=-l}^l \lambda_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} \eta_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}, \\ \psi &= \sum_{l=0} \sum_{m=-l-1}^l \sum_{\tilde{m}=-l}^l \psi_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} \eta_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} + \sum_{l=1/2}^{l-1} \sum_{m=-l}^{l-1} \sum_{\tilde{m}=-l}^l \psi_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}} \eta_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}},\end{aligned}$$

into the Lagrangian $\mathcal{L}_{VQ}^{(F)}$, one obtains the action $S_{VQ}^{(F)} = \int_{S^3} \mathcal{L}_{VQ}^{(F)} * 1$. There one finds the terms including

$$\int_{S^3} (\eta_{J', M'; l', \tilde{m}'}^\dagger) (1 - k_m \gamma^m) \eta_{J, M; l, \tilde{m}} * 1 = \langle \langle J', M' | 1 - 2\mathcal{S}^3 | J, M \rangle \rangle \delta_{l, l'} \delta_{m, m'},$$

with our choice $k^a = \delta_3^a$. For the coefficients $\langle \langle J', M' | 1 - 2\mathcal{S}^3 | J, M \rangle \rangle$, see appendix B.

Similarly to the bosonic part, the modes $\lambda_{l\pm\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}, \psi_{l\pm\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}$ in each sector $(\alpha, l, m, \tilde{m})$ decouple from the modes in the other sectors, and therefore the action $S_{VQ}^{(F)}$ can be divided into the actions $S_{VQ; \alpha, l, m, \tilde{m}}^{(F)}$ of each sector (l, m, \tilde{m}) as

$$S_{VQ}^{(F)} = \sum_{\alpha \in \Lambda} \sum_{l=0}^{\infty} \sum_{m=-(l+1)}^l \sum_{\tilde{m}=-l}^l S_{VQ; \alpha, l, m, \tilde{m}}^{(F)} + S_{VQ; H}^{(F)},$$

where $S_{VQ; H}^{(F)}$ consists of the modes in the Cartan subalgebra of G .

After completing the square and shifting the fields properly, the action $S_{VQ; \alpha, l, m, \tilde{m}}^{(F)}$ of the modes for $l \geq 1/2, -l \leq m \leq (l-1), -l \leq \tilde{m} \leq l$, is given by

$$\begin{aligned}& \left(\left(\psi_{z; l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger, \left(\psi_{z; l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger \right) g^{\bar{z}z} \mathcal{K}_{l, m}^\alpha \begin{pmatrix} \psi_{z; l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \\ \psi_{z; l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \end{pmatrix} \\ & + \left(\left(\lambda_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger, \left(\lambda_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger \right) \mathcal{M}_{l, m}^\alpha \begin{pmatrix} \lambda_{l+\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \\ \lambda_{l-\frac{1}{2}, m+\frac{1}{2}; l, \tilde{m}}^\alpha \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\mathcal{K}_{l,m}^\alpha &= \begin{pmatrix} -\frac{2l(l+1)-m}{2l+1} + \frac{i}{2}g(\alpha \cdot \sigma) & -\frac{1}{2l+1}\sqrt{(l+m+1)(l-m)} \\ -\frac{1}{2l+1}\sqrt{(l+m+1)(l-m)} & \frac{2l(l+1)-m}{2l+1} + \frac{i}{2}g(\alpha \cdot \sigma) \end{pmatrix}, \\ \mathcal{M}_{l,m}^\alpha &= \begin{pmatrix} (2(l+1) + ig(\alpha \cdot \sigma)) & \\ & (-2l + ig(\alpha \cdot \sigma)) \end{pmatrix} \\ &\quad \times \begin{pmatrix} -\frac{l-m}{2l+1} \frac{2}{K_{l,m}^\alpha} g^{\bar{z}z} D_{\bar{z}} D_z + 1 & -\frac{\sqrt{(l+m+1)(l-m)}}{2l+1} \frac{2}{K_{l,m}^\alpha} g^{\bar{z}z} D_{\bar{z}} D_z \\ -\frac{\sqrt{(l+m+1)(l-m)}}{2l+1} \frac{2}{K_{l,m}^\alpha} g^{\bar{z}z} D_{\bar{z}} D_z & -\frac{l+m+1}{2l+1} \frac{2}{K_{l,m}^\alpha} g^{\bar{z}z} D_{\bar{z}} D_z + 1 \end{pmatrix},\end{aligned}$$

and one can see that to the one-loop determinant, they yield the contributions

$$\begin{aligned}\prod_{\alpha \in \Lambda} \prod_{l=1/2}^{\infty} \prod_{m=-l}^{l-1} \prod_{\tilde{m}=-l}^l \frac{\text{Det}_{(1,0)} [K_{l,m}^\alpha]}{\text{Det}_{(0,0)} [K_{l,m}^\alpha]} \text{Det}_{(0,0)} [2g^{\bar{z}z} D_{\bar{z}} D_z - K_{l,m}^\alpha] \\ \times \text{Det}_{(0,0)} [(2(l+1) + ig(\alpha \cdot \sigma)) (-2l + ig(\alpha \cdot \sigma))],\end{aligned}$$

up to an overall normalization constant.

For the remaining fermionic modes with $l \geq 0$, $m = -(l+1), l$; $-l \leq \tilde{m} \leq l$, after some similar algebra, one obtains

$$\begin{aligned}& \left(\lambda_{l+\frac{1}{2}, l+\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger [2(l+1) + ig(\alpha \cdot \sigma)] \lambda_{l+\frac{1}{2}, l+\frac{1}{2}; l, \tilde{m}}^\alpha \\ & + \left(\lambda_{l+\frac{1}{2}, -l-\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger \frac{1}{-2(l+1) + ig(\alpha \cdot \sigma)} [2g^{\bar{z}z} D_{\bar{z}} D_z - K_{l, -l-1}^\alpha] \lambda_{l+\frac{1}{2}, -l-\frac{1}{2}; l, \tilde{m}}^\alpha \\ & + \frac{1}{2} g^{\bar{z}z} \left(\psi_{z; l+\frac{1}{2}, l+\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger [-2l + ig(\alpha \cdot \sigma)] \psi_{z; l+\frac{1}{2}, l+\frac{1}{2}; l, \tilde{m}}^\alpha \\ & + \frac{1}{2} g^{\bar{z}z} \left(\psi_{z; l+\frac{1}{2}, -l-\frac{1}{2}; l, \tilde{m}}^\alpha \right)^\dagger [-2(l+1) + ig(\alpha \cdot \sigma)] \psi_{z; l+\frac{1}{2}, -l-\frac{1}{2}; l, \tilde{m}}^\alpha,\end{aligned}$$

and finds the one-loop determinant

$$\begin{aligned}\prod_{\alpha \in \Lambda} \prod_{l=0}^{\infty} \prod_{\tilde{m}=-l}^l \text{Det}_{(1,0)} [2(l+1) + ig(\alpha \cdot \sigma)] \text{Det}_{(1,0)} [-2l + ig(\alpha \cdot \sigma)] \\ \times \text{Det}_{(0,0)} [-2g^{\bar{z}z} D_{\bar{z}} D_z + K_{l, -l-1}^\alpha],\end{aligned}$$

up to an overall constant.

Wrapping up the contributions from the bosonic fluctuations and the fermionic fluctuations to the one-loop determinant, one obtains

$$\prod_{\alpha \in \Lambda_+} \frac{1}{\text{Det}_{(1,0)} [K_{0,0}^\alpha]} \prod_{l=\frac{1}{2}}^{\infty} \left(\frac{\text{Det}_{(0,0)} [K_{l,l}^\alpha]}{\text{Det}_{(1,0)} [K_{l,l}^\alpha]} \right)^2,$$

up to an overall normalization constant. Taking account of this result and the one-loop determinant (24) from the ghost, and using the same reason of the Hodge decomposition as in [19], one finds that the total one-loop determinant is given by

$$\left(\prod_{\alpha \in \Lambda_+} \sin(i\pi g(\alpha \cdot \sigma)) \right)^{\chi(\Sigma)}. \quad (27)$$

This is one of the main results in this paper.

There are two subtle points related to the zero modes in the Cartan subalgebra, on which so far we have not discussed in detail. In the Cartan part, there is the fermion zero modes $(\psi_{z;\frac{1}{2},\frac{1}{2},0,0}^i)^\dagger$, $\psi_{z;\frac{1}{2},\frac{1}{2},0,0}^i$, but the term $g^{\bar{z}z}\bar{\psi}_z\psi_z$ in the Lagrangian \mathcal{L}'_V absorbs them. Therefore, they cause no problems.

We are left to perform the path integral over the background gauge fields $v_z^i(z, \bar{z})$ ($i = 1, \dots, r$), along with the finite dimensional integral over the background σ^i ($i = 1, \dots, r$). While the background of the gauge fields $v_z^i(z, \bar{z})$ obeying $\int_\Sigma v_z^i d\bar{z} \wedge dz \neq 0$ in the classical action \mathcal{L}_{cl} can contribute to the path integral, Upon the integration over the gauge fields $v_z^i(z, \bar{z})$, the fluctuations of the gauge fields around the background don't appear in the rest of the path integral. One therefore needs to divide the integration over the fluctuations, along with the other possible constant factors.

6.2 The Contribution from the Hypermultiplet

Let us proceed to the hypermultiplet. Along with the BRST transformation (16) of the vector multiplet, the hypermultiplet transform under the BRST transformation as

$$\begin{aligned}
\delta_Q \tilde{H} &= 0, & \delta_Q H &= 0, \\
\delta_Q (\tilde{H})^* &= -i (\tilde{\chi})^\dagger \epsilon, & \delta_Q (H)^* &= -i (\chi)^\dagger \epsilon, \\
\delta_Q \tilde{\chi} &= [D_m \tilde{H} \gamma^m - g [\sigma, \tilde{H}] + i \tilde{H}] \epsilon, & \delta_Q (\tilde{\chi})^\dagger &= \epsilon^T C_3 [2i D_z H + (F_{H2})^*], \\
\delta_Q \chi &= [D_m H \gamma^m - g [\sigma, H] + i H] \epsilon, & \delta_Q (\chi)^\dagger &= -\epsilon^T C_3 [2i D_z \tilde{H} + F_{H1}], \\
\delta_Q F_{H1} &= 0, & \delta_Q (F_{H2})^* &= 0, \\
\delta_Q (F_{H1})^* &= i \left[-D_m \chi^T C_3 \gamma^m - 2i (D_z \tilde{\chi})^\dagger - \frac{i}{2} \chi^T C_3 \right. \\
&\quad \left. + g [\sigma, \chi^T] C_3 - 2ig [\tilde{H}^*, \psi^\dagger] - 2i [H, \lambda^T] C_3 \right] \epsilon, \\
\delta_Q F_{H2} &= i \epsilon^T C_3 \left[\gamma^m D_m \tilde{\chi} - 2i C_3^{-1} (D_z \chi)^* - \frac{i}{2} \tilde{\chi} \right. \\
&\quad \left. + g [\sigma, \tilde{\chi}] - 2ig [\tilde{H}, \lambda] - 2i C_3^{-1} [H^*, \psi^*] \right].
\end{aligned} \tag{28}$$

Note that the analytic continuation for the scalar field σ has already been done here.

In order to carry out the localization procedure, we will add the Lagrangian

$$\mathcal{L}_{HQ} = \delta_Q \left[(\delta_Q \tilde{\chi})^\dagger \tilde{\chi} + (\tilde{\chi})^\dagger (\delta_Q (\tilde{\chi})^\dagger)^\dagger + (\delta_Q \chi)^\dagger \chi + (\chi)^\dagger (\delta_Q (\chi)^\dagger)^\dagger \right], \tag{29}$$

to the Lagrangian \mathcal{L}_{VQ} in (17). The total Lagrangian \mathcal{L} will thus be shifted as $\mathcal{L} \rightarrow \mathcal{L} - t(\mathcal{L}_{VQ} + \mathcal{L}_{HQ})$.

A fixed point is given by a solution to $\delta_Q \chi = 0$, $\delta_Q \tilde{\chi} = 0$ meaning that

$$D_m \tilde{H} \gamma^m + i \tilde{H} - g [\sigma, \tilde{H}] = 0, \quad D_m H \gamma^m + i H - g [\sigma, H] = 0,$$

and to $\delta_Q \tilde{\chi}^\dagger = 0$, $\delta_Q \tilde{\chi}^\dagger = 0$. Since the solution to the former equations is given by $\tilde{H} = 0$, $H = 0$, substituting it into the latter equations, one obtains the solution $F_{H1} = 0$, $F_{H2} = 0$. One thus finds no non-trivial backgrounds.

Then, up to quadratic order of the fluctuations, the bosonic part $\mathcal{L}_{HQ}^{(B)}$ of the Lagrangian \mathcal{L}_{HQ} is given by

$$\begin{aligned} & \text{tr} \left[\left(D_m \tilde{H} \right)^\dagger D^m \tilde{H} + (D_m H)^\dagger D^m H \right. \\ & + \left(\tilde{H} + ig [\sigma, \tilde{H}] \right)^\dagger \left(\tilde{H} + ig [\sigma, \tilde{H}] \right) + (H + ig [\sigma, H])^\dagger (H + ig [\sigma, H]) \\ & \left. + \left(F_{H1} + 2i D_z \tilde{H} \right)^\dagger \left(F_{H1} + 2i D_z \tilde{H} \right) + \left(F_{H2} - 2i \left(D_z \tilde{H} \right)^* \right)^\dagger \left(F_{H2} - 2i \left(D_z \tilde{H} \right)^* \right) \right], \end{aligned}$$

where σ is the fixed point (22), and the fermionic part $\mathcal{L}_{HQ}^{(F)}$ by

$$\text{tr} \left[\tilde{\chi}^\dagger k_n \gamma^n \left(i \gamma^m D_m \tilde{\chi} + \frac{1}{2} \tilde{\chi} - ig [\sigma, \tilde{\chi}] \right) + \chi^\dagger k_n \gamma^n \left(i \gamma^m D_m \chi + \frac{1}{2} \chi - ig [\sigma, \chi] \right) \right].$$

We will carry out similar calculations to what we have done for the vector multiplet by substituting the spherical harmonic expansions of the fluctuations

$$\begin{aligned} \tilde{H} &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\tilde{m}=-l}^l \tilde{H}_{l,m,\tilde{m}} \varphi_{l,m,\tilde{m}}, & H &= \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\tilde{m}=-l}^l H_{l,m,\tilde{m}} \varphi_{l,m,\tilde{m}}, \\ \tilde{\chi} &= \sum_{l=0}^{\infty} \sum_{m=-l-1}^l \sum_{\tilde{m}=-l}^l \tilde{\chi}_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} + \sum_{l=1/2}^{\infty} \sum_{m=-l}^{l-1} \sum_{\tilde{m}=-l}^l \tilde{\chi}_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}, \\ \chi &= \sum_{l=0}^{\infty} \sum_{m=-l-1}^l \sum_{\tilde{m}=-l}^l \chi_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \eta_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} + \sum_{l=1/2}^{\infty} \sum_{m=-l}^{l-1} \sum_{\tilde{m}=-l}^l \chi_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \eta_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}, \end{aligned}$$

into \mathcal{L}_{HQ} . Recalling that $k_n \gamma^n = 2\mathcal{S}_3$ and using the Clebsch-Gordan coefficients in appendix B, one obtains the root part of $\mathcal{L}_{HQ}^{(B)}$

$$\sum_{\alpha \in \Lambda} \sum_{l=0}^{\infty} \sum_{m=-l}^l \sum_{\tilde{m}=-l}^l [4l(l+1) + 1 + g^2(\alpha \cdot \sigma)^2] \left(\left| \tilde{H}_{l,m,\tilde{m}}^\alpha \right|^2 + \left| H_{l,m,\tilde{m}}^\alpha \right|^2 \right), \quad (30)$$

and the root part of $\mathcal{L}_{HQ}^{(F)}$ is given by the sum of

$$\begin{aligned} & \sum_{\alpha \in \Lambda} \left\{ \sum_{l=0}^{\infty} \sum_{\tilde{m}=-l}^l \left[-\tilde{\chi}_{l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}}^\dagger [(2l+1) + ig(\alpha \cdot \sigma)] \tilde{\chi}_{l+\frac{1}{2},l+\frac{1}{2};l,\tilde{m}} \right. \right. \\ & \quad \left. \left. + \tilde{\chi}_{l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}}^\dagger [(2l+1) + ig(\alpha \cdot \sigma)] \tilde{\chi}_{l+\frac{1}{2},-l-\frac{1}{2};l,\tilde{m}} \right] \right. \\ & \quad \left. - \sum_{l=1/2}^{\infty} \sum_{m=-l}^{l-1} \sum_{\tilde{m}=-l}^l \left[\left(\tilde{\chi}_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}^\dagger, \tilde{\chi}_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}}^\dagger \right) \begin{pmatrix} \mathcal{A}_{l,m}^\alpha & \mathcal{B}_{l,m}^\alpha \\ \mathcal{C}_{l,m}^\alpha & \mathcal{D}_{l,m}^\alpha \end{pmatrix} \begin{pmatrix} \tilde{\chi}_{l+\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \\ \tilde{\chi}_{l-\frac{1}{2},m+\frac{1}{2};l,\tilde{m}} \end{pmatrix} \right] \right\}, \end{aligned} \quad (31)$$

and the same terms with $\tilde{\chi}$'s replaced by χ 's, where

$$\begin{aligned}\mathcal{A}_{l,m}^\alpha &= \frac{2m+1}{2l+1} ((2l+1) + ig(\alpha \cdot \sigma)), \\ \mathcal{B}_{l,m}^\alpha &= -2 \frac{\sqrt{(l+m+1)(l-m)}}{2l+1} ((2l+1) + ig(\alpha \cdot \sigma)), \\ \mathcal{C}_{l,m}^\alpha &= 2 \frac{\sqrt{(l+m+1)(l-m)}}{2l+1} ((2l+1) - ig(\alpha \cdot \sigma)), \\ \mathcal{D}_{l,m}^\alpha &= \frac{2m+1}{2l+1} ((2l+1) - ig(\alpha \cdot \sigma)).\end{aligned}$$

Note that the integration over the auxiliary fields F_{H1} , F_{H2} has already been done, and their contributions to the partition function is just an overall constant.

Although the Cartan part of \mathcal{L}_{HQ} can be easily obtained by setting α to zero and by replacing the sum over the Λ by over i running from 1 to r , we aren't interested in the overall normalization constant of the partition function, to which the Cartan part can only contribute. Therefore, we will focus on the root part, as for the vector multiplet.

From (30) and (31), one can easily see that the one-loop determinant from the bosonic fluctuations yields

$$\prod_{\alpha \in \Lambda_+} \prod_{l=0}^{\infty} \prod_{m=-l}^l \prod_{\tilde{m}=-l}^l \left(\frac{1}{\text{Det}_{(\frac{1}{2},0)} [(2l+1)^2 + g^2(\alpha \cdot \sigma)^2]} \right)^2,$$

and the one from the fermionic fluctuations,

$$\prod_{\alpha \in \Lambda_+} \prod_{l=0}^{\infty} \prod_{m=-l}^l \prod_{\tilde{m}=-l}^l \left(\text{Det}_{(\frac{1}{2},0)} [(2l+1)^2 + g^2(\alpha \cdot \sigma)^2] \right)^2,$$

up to an overall constant.

Wrapping up them, one finds that the hypermultiplet contributes just a constant to the total partition function.

7 Discussions

From (27) and (19), the partition function of both of the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories reduce to the finite dimensional integral

$$Z_{5DSYM} = \mathcal{N}_{5DSYM} \sum_m \prod_{i=1}^r \int d\sigma^i \left[\prod_{\alpha \in \Lambda_+} 2 \sin(i\pi g(\alpha \cdot \sigma)) \right]^{\chi(\Sigma)} \exp \left[\int \mathcal{L}_{YM} d(\text{vol}) \right], \quad (32)$$

with m the first Chern number of the two-dimensional gauge field v_z on Σ , where the normalization constant \mathcal{N}_{5DSYM} may be different in the $\mathcal{N} = 2$ theory from in the $\mathcal{N} = 1$ theory.

Let us find the parameter q from (32), and furthermore, for the comparison¹⁰ with the prediction from the conjecture [4, 5] for the six-dimensional $\mathcal{N} = (2, 0)$ theory, we will replace the radius of the unit S^3 by l . For brevity, we will take the $SU(2)$ gauge group.

¹⁰We would like to thank Yuji Tachikawa for suggesting us to check the consistency of our results with the conformal index in [1].

Then, the result (27)

$$\left[\prod_{\alpha \in \Lambda_+} 2 \sin(i\pi g(\alpha \cdot \sigma)) \right]^{\chi(\Sigma)}$$

reduces into

$$\left[2 \sin\left(i\sqrt{2}\pi gl\sigma\right) \right]^{\chi(\Sigma)}. \quad (33)$$

From the Lagrangian (19)

$$\mathcal{L}_{YM} = 2\pi^2 l^3 \text{tr} \left[-\left(\frac{\sigma}{l}\right)^2 + 2\frac{\sigma}{l} g^{\bar{z}z} v_{\bar{z}z} \right],$$

the classical action is given by

$$\int_{\Sigma} \mathcal{L}_{YM} d(\text{vol}) = -2\pi^2 l \int d(\text{vol}) (\sigma)^2 - 4i\pi^2 l^2 \sigma \int_{\Sigma} v_{\bar{z}z} d\bar{z} \wedge dz.$$

As explained in detail in [19], we need the summation over the first Chern numbers of the two-dimensional gauge field $v_z, v_{\bar{z}}$ on Σ . Here, let us explain the fact that the normalization of the first Chern number is given by

$$\int_{\Sigma} v_{\bar{z}z} d\bar{z} \wedge dz = \frac{\sqrt{2}}{g} 2\pi m, \quad (34)$$

with $m \in \mathbb{Z}$. Let us recall that the Cartan subalgebra of the $SU(2)$ gauge group is generated by $H = \sigma_3/\sqrt{2}$, with the normalization $\text{tr}[HH] = 1$. The Lie algebra of the $SU(2)$ gauge group in fact is generated by H and E_{\pm} , which obey that

$$[H, E_{\pm}] = \pm\sqrt{2}E_{\pm}, \quad [E_+, E_-] = \sqrt{2}H,$$

in our convention. Therefore, a field ψ in the fundamental representation of the gauge group can be decomposed into the eigenstates of H as

$$H\psi_{\pm} = \pm\frac{1}{\sqrt{2}}\psi_{\pm},$$

and the covariant derivative gives

$$D\psi_+ = d\psi_+ + \frac{i}{\sqrt{2}}g v\psi_+.$$

Under a gauge transformation, the two-dimensional gauge field v_z transforms in a differential form notation as

$$v \rightarrow v - \frac{\sqrt{2}}{g} d\Omega,$$

and then the field ψ_+ transforms as

$$\psi_+ \rightarrow e^{i\Omega}\psi_+.$$

For brevity, let us take $\Sigma = S^2$ and consider two patches $U_N = \{(\theta, \phi) | 0 \leq \theta \leq \pi/2\}$ and $U_S = \{(\theta, \phi) | \pi/2 \leq \theta \leq \pi\}$, covering the S^2 with the polar coordinates (θ, ϕ) . On $U_N \cap U_S$, suppose that the section ψ_N of ψ_+ on U_N is related to the section ψ_S on U_S as $\psi_S = e^{i\Omega} \psi_N$. Then, the requirement that ψ_S be single-valued is satisfied if

$$\Omega = m\phi, \quad (m \in \mathbb{Z})$$

on the $U_N \cap U_S$. Then, the flux is determined as

$$\int_{\Sigma} dv = \int_{U_N \cap U_S} (v_N - v_S) = \int_{U_N \cap U_S} \frac{\sqrt{2}}{g} d\Omega = \frac{\sqrt{2}}{g} (2\pi m).$$

Substituting the gauge field configurations (34) into the partition function (32) and summing up over the Chern number m , one can see that the dominant contribution from the integration over σ is given by the points

$$\sigma = \frac{g}{\sqrt{2}} \frac{n}{4\pi^2 l^2},$$

with $n \in \mathbb{Z}$. Since the measure (33) at the dominant points of σ yields

$$\left(2 \sin \left(i \frac{g^2}{4\pi l} n \right) \right)^{\chi(\Sigma)} = \left[e^{-\frac{g^2}{4\pi l} \cdot n} - e^{+\frac{g^2}{4\pi l} \cdot n} \right]^{\chi(\Sigma)} = [n]_q,$$

we obtain the parameter

$$q = \exp\left(-\frac{g^2}{2\pi l}\right). \quad (35)$$

In the paper [1], the superconformal index in four-dimensional $\mathcal{N} = 2$ gauge theories was calculated on $S^3 \times S^1$. In the index, one can see that the parameter q is found in the form

$$q^{\Delta} = e^{-\beta E},$$

where Δ is the conformal weight of states over which the index have the summation, and the energy E can be obtained through the state-operator mapping in conformal field theories. The temperature β is the radius of S^1 , which we regard as the circle on which the six-dimensional $\mathcal{N} = (2, 0)$ theory is placed to yield the five-dimensional $\mathcal{N} = 2$ theory.

Following [4, 5], instanton solutions in the five-dimensional $\mathcal{N} = 2$ theory correspond to the Kaluza-Klein modes in the six-dimensional $\mathcal{N} = (2, 0)$ theory compactified on S^1 . In a four-dimensional $SU(2)$ gauge theory, the one-instanton solution gives the classical action

$$\int d^4x F_{mn}^a F_{mn}^a = 32\pi^2,$$

where $m, n = 1, \dots, 4$, with the normalization

$$F_{mn}^a = \partial_m A_n^a - \partial_n A_m^a + \epsilon^{abc} A_m^b A_n^c.$$

Therefore, for our convention, identifying

$$v_m^a = -\frac{1}{\sqrt{2}g}A_m^a, \quad (m = 1, \dots, 4)$$

we can see that

$$v_{mn}^a = \partial_m v_n^a - \partial_n v_m^a - \sqrt{2}g\epsilon^{abc}v_m^b v_n^c = -\frac{1}{\sqrt{2}g}F_{mn}^a,$$

where $m, n = 1, \dots, 4$, and thus in the five-dimensional theory, one finds the classical action

$$\int d^5X - \frac{1}{4}v_{MN}^a v_{MN}^a = -\frac{4\pi^2}{g^2} \int dX_5$$

for the instanton solution of unit instanton charge.

For the six-dimensional $(2, 0)$ theory on S^1 of radius R , the instanton solution of unit instanton charge corresponds to the first KK modes, and so one obtains the relation

$$\frac{1}{R} = \frac{4\pi^2}{g^2}.$$

One thus finds the temperature

$$\beta = 2\pi R = \frac{g^2}{2\pi}. \quad (36)$$

For a 4-dimensional massless scalar of conformal weight $\Delta = 1$, the conformal coupling term of it with the scalar curvature in the Lagrangian gives it a mass

$$E = m = \frac{1}{l},$$

on $\mathbb{R} \times S^3$, where the radius of the S^3 is l . Therefore, the state-operator mapping in conformal field theories suggests that

$$E = \frac{\Delta}{l}. \quad (37)$$

Wrapping up (36) and (37) to obtain

$$e^{-\beta E} = e^{-\frac{g^2}{2\pi} \frac{\Delta}{l}} = e^{-\frac{g^2}{2\pi l} \Delta},$$

the parameter q can now be read as

$$q = e^{-\frac{g^2}{2\pi l}},$$

which is in perfect agreement with our result in the five-dimensional theory.

Thus, we have seen that the partition function of five-dimensional theory yields the partition function of the two-dimensional q -deformed Yang-Mills theory, but not the ordinary Yang-Mills theory on a closed Riemann surface. It is consistent with the proposal in [1].

Furthermore, in order for the parameter q found in the result of the five-dimensional theory to be identical to the q in the conformal index of [1], we must identify the five-dimensional gauge coupling constant g as the temperature β or equivalently the radius R of the S^1 in the four-dimensional theory. However, the identification is also consistent with the prediction of the conjecture [4, 5] that instanton solutions in the five-dimensional $\mathcal{N} = 2$ theory is identical to the Kaluza-Klein modes in the six-dimensional $\mathcal{N} = (2, 0)$ theory.

Since the hypermultiplet gives no contributions to these results, what we have discussed above is also the case for the five-dimensional $\mathcal{N} = 1$ theory. However, the existence of the corresponding six-dimensional theory is not clear for us up to this point.

Recently, the authors of [23] and of [24] have considered superconformal indices of four-dimensional $\mathcal{N} = 2$ gauge theories with more parameters, and found that the indices give rise to not just the q -deformed two-dimensional Yang-Mills theory, but the (q, t) -deformed and the (p, q, t) -deformed ones. It would be interesting to extend the localization analysis in this paper to these cases.

Acknowledgement

The authors would like to thank Kazuo Hosomichi for helpful discussions. We are also grateful to Yuji Tachikawa for helpful discussions, valuable suggestions and a careful reading of the manuscript. The work of T. K. was supported in part by a Grant-in-Aid #23540286 from the MEXT of Japan.

A Gamma Matrices

The five-dimensional gamma matrices Γ^M ($M = 1, \dots, 5$) satisfy

$$\{\Gamma^M, \Gamma^N\} = 2\delta^{MN},$$

and they are given in terms of the three-dimensional gamma matrices $\gamma^m = \sigma_m$ ($m = 1, 2, 3$) as

$$\Gamma^m = \gamma^m \otimes \sigma_2, \quad \Gamma^4 = \mathbf{1} \otimes \sigma_1, \quad \Gamma^5 = \mathbf{1} \otimes \sigma_3,$$

where $\sigma_{1,2,3}$ are the Pauli matrices.

The five-dimensional charge conjugation matrix C_5 satisfies

$$(\Gamma^M)^T = C_5 \Gamma^M C_5^{-1}, \quad (C_5)^T = -C_5,$$

where T denotes the transpose of the matrix, and it may be given in terms of the three-dimensional charge conjugate matrix $C_3 = i\sigma_2$ as

$$C_5 = C_3 \otimes \mathbf{1}.$$

B The Clebsch-Gordan Coefficients

B.1 The spin l representation \otimes the spin 1 representation into the spin $J = l, l \pm 1$ representations

- the spin $J = l + 1$ representation

$$\begin{aligned} & |J = l + 1, M = m\rangle\rangle \\ &= \sqrt{\frac{1}{2(l+1)(2l+1)}} \left[\sqrt{(l+m)(l+m+1)} |l, m-1\rangle |1, 1\rangle \right. \\ & \quad \left. + \sqrt{2(l+m+1)(l-m+1)} |l, m\rangle |1, 0\rangle + \sqrt{(l-m)(l-m+1)} |l, m+1\rangle |1, -1\rangle \right]. \end{aligned}$$

- the spin $J = l$ representation

$$\begin{aligned} & |J = l, M = m\rangle\rangle \\ &= \sqrt{\frac{1}{2l(l+1)}} \left[-\sqrt{(l+m)(l-m+1)} |l, m-1\rangle |1, 1\rangle \right. \\ & \quad \left. + \sqrt{2m} |l, m\rangle |1, 0\rangle + \sqrt{(l-m)(l+m+1)} |l, m+1\rangle |1, -1\rangle \right]. \end{aligned}$$

- the spin $J = l - 1$ representation

$$\begin{aligned} & |J = l - 1, M = m\rangle\rangle \\ &= \sqrt{\frac{1}{2l(2l+1)}} \left[\sqrt{(l-m)(l-m+1)} |l, m-1\rangle |1, 1\rangle \right. \\ & \quad \left. - \sqrt{2(l+m)(l-m)} |l, m\rangle |1, 0\rangle + \sqrt{(l+m)(l+m+1)} |l, m+1\rangle |1, -1\rangle \right]. \end{aligned}$$

Furthermore, the action of the spin operator S_3 on the state $|J, M\rangle\rangle$ is obtained as

$$\begin{aligned}
& S_3 |l+1, M\rangle\rangle \\
&= \frac{M}{l+1} |l+1, M\rangle\rangle - \frac{1}{l+1} \sqrt{\frac{l}{2l+1}} \sqrt{(l+M+1)(l-M+1)} |l, M\rangle\rangle, \\
& S_3 |l, M\rangle\rangle \\
&= \sqrt{\frac{1}{l(l+1)}} \left[-l \sqrt{\frac{(l-M+1)(l+M+1)}{(l+1)(2l+1)}} |l+1, M\rangle\rangle \right. \\
&\quad \left. + M \sqrt{\frac{1}{l(2l+1)}} |l, M\rangle\rangle - (l+1) \sqrt{\frac{(l-M)(l+M)}{l(2l+1)}} |l-1, M\rangle\rangle \right], \\
& S_3 |l-1, M\rangle\rangle \\
&= -\frac{M}{l} |l-1, M\rangle\rangle - \frac{1}{l} \sqrt{\frac{l+1}{2l+1}} \sqrt{(l+M)(l-M)} |l, M\rangle\rangle.
\end{aligned}$$

B.2 The spin l representation \otimes the spin $1/2$ representation into the spin $J = l \pm 1/2$ representations

- the spin $J = l + 1/2$ representation ($m = -l-1, -l, \dots, l-1, l$)

$$|J = l + \frac{1}{2}, M = m + \frac{1}{2}\rangle\rangle = \sqrt{\frac{l-m}{2l+1}} |l, m+1\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle + \sqrt{\frac{l+m+1}{2l+1}} |l, m\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle.$$

- the spin $J = l - 1/2$ representation ($m = -l, \dots, l-1$)

$$|J = l - \frac{1}{2}, M = m + \frac{1}{2}\rangle\rangle = \sqrt{\frac{l+m+1}{2l+1}} |l, m+1\rangle \left| \frac{1}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{l-m}{2l+1}} |l, m\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle.$$

Furthermore, the action of the spin operator S_3 on the state $|J, M\rangle\rangle$ is obtained as

$$\begin{aligned}
& S_3 |l + \frac{1}{2}, m + \frac{1}{2}\rangle\rangle \\
&= \frac{1}{2} \frac{2m+1}{2l+1} |l + \frac{1}{2}, m + \frac{1}{2}\rangle\rangle - \frac{\sqrt{(l+m+1)(l-m)}}{2l+1} |l - \frac{1}{2}, m + \frac{1}{2}\rangle\rangle, \\
& S_3 |l - \frac{1}{2}, m + \frac{1}{2}\rangle\rangle \\
&= -\frac{\sqrt{(l+m+1)(l-m)}}{2l+1} |l + \frac{1}{2}, m + \frac{1}{2}\rangle\rangle - \frac{1}{2} \frac{2m+1}{2l+1} |l - \frac{1}{2}, m + \frac{1}{2}\rangle\rangle.
\end{aligned}$$

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